Your Turn

A sporting goods store sells reusable sports water bottles for $8. At this price their weekly sales are approximately 100 items. Research says that for every $2 increase in price, the manager can expect the store to sell five fewer water bottles.

a) Represent this situation with a quadratic function.
b) Determine the maximum revenue the manager can expect based on these estimates. What selling price will give that maximum revenue?
c) Verify your solution.
d) Explain any assumptions you made in using a quadratic function in this situation.

Key Ideas

- You can convert a quadratic function from standard form to vertex form by completing the square.

$y = 5x^2 - 30x + 7$	← standard form
$y = 5(x^2 - 6x) + 7$	Group the first two terms. Factor out the leading coefficient if $a \neq 1$.
$y = 5(x^2 - 6x + 9 - 9) + 7$	Add and then subtract the square of half the coefficient of the x-term.
$y = 5[(x^2 - 6x + 9) - 9] + 7$	Group the perfect square trinomial.
$y = 5[(x - 3)^2 - 9] + 7$	Rewrite using the square of a binomial.
$y = 5(x - 3)^2 - 45 + 7$	Simplify.
$y = 5(x - 3)^2 - 38$	← vertex form

- Converting a quadratic function to vertex form, $y = a(x - p)^2 + q$, reveals the coordinates of the vertex, (p, q).
- You can use information derived from the vertex form to solve problems such as those involving maximum and minimum values.

Check Your Understanding

Practise

1. Use a model to determine the value of c that makes each trinomial expression a perfect square. What is the equivalent binomial square expression for each?

a) $x^2 + 6x + c$

b) $x^2 - 4x + c$

c) $x^2 + 14x + c$

d) $x^2 - 2x + c$

2. Write each function in vertex form by completing the square. Use your answer to identify the vertex of the function.

a) $y = x^2 + 8x$

b) $y = x^2 - 18x - 59$

c) $y = x^2 - 10x + 31$

d) $y = x^2 + 32x - 120$

3. Convert each function to the form $y = a(x - p)^2 + q$ by completing the square. Verify each answer with or without technology.

a) $y = 2x^2 - 12x$

b) $y = 6x^2 + 24x + 17$

c) $y = 10x^2 - 160x + 80$

d) $y = 3x^2 + 42x - 96$

4. Convert each function to vertex form algebraically, and verify your answer.

a) $f(x) = -4x^2 + 16x$

b) $f(x) = -20x^2 - 400x - 243$

c) $f(x) = -x^2 - 42x + 500$

d) $f(x) = -7x^2 + 182x - 70$

5. Verify, in at least two different ways, that the two algebraic forms in each pair represent the same function.

a) $y = x^2 - 22x + 13$
and
$y = (x - 11)^2 - 108$

b) $y = 4x^2 + 120x$
and
$y = 4(x + 15)^2 - 900$

c) $y = 9x^2 - 54x - 10$
and
$y = 9(x - 3)^2 - 91$

d) $y = -4x^2 - 8x + 2$
and
$y = -4(x + 1)^2 + 6$

6. Determine the maximum or minimum value of each function and the value of x at which it occurs.

a) $y = x^2 + 6x - 2$

b) $y = 3x^2 - 12x + 1$

c) $y = -x^2 - 10x$

d) $y = -2x^2 + 8x - 3$

7. For each quadratic function, determine the maximum or minimum value.

a) $f(x) = x^2 + 5x + 3$

b) $f(x) = 2x^2 - 2x + 1$

c) $f(x) = -0.5x^2 + 10x - 3$

d) $f(x) = 3x^2 - 4.8x$

e) $f(x) = -0.2x^2 + 3.4x + 4.5$

f) $f(x) = -2x^2 + 5.8x - 3$

8. Convert each function to vertex form.

a) $y = x^2 + \frac{3}{2}x - 7$

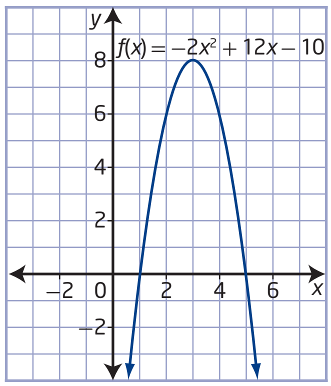
b) $y = -x^2 - \frac{3}{8}x$

c) $y = 2x^2 - \frac{5}{6}x + 1$

Apply

9. a) Convert the quadratic function $f(x) = -2x^2 + 12x - 10$ to vertex form by completing the square.

b) The graph of $f(x) = -2x^2 + 12x - 10$ is shown. Explain how you can use the graph to verify your answer.



10. a) For the quadratic function $y = -4x^2 + 20x + 37$, determine the maximum or minimum value and domain and range without making a table of values or graphing.

b) Explain the strategy you used in part a).

11. Determine the vertex of the graph of $f(x) = 12x^2 - 78x + 126$. Explain the method you used.

12. Identify, explain, and correct the error(s) in the following examples of completing the square.

a) $y = x^2 + 8x + 30$
$y = (x^2 + 4x + 4) + 30$
$y = (x + 2)^2 + 30$

b) $f(x) = 2x^2 - 9x - 55$
$f(x) = 2(x^2 - 4.5x + 20.25 - 20.25) - 55$
$f(x) = 2[(x^2 - 4.5x + 20.25) - 20.25] - 55$
$f(x) = 2[(x - 4.5)^2 - 20.25] - 55$
$f(x) = 2(x - 4.5)^2 - 40.5 - 55$
$f(x) = (x - 4.5)^2 - 95.5$

c) $y = 8x^2 + 16x - 13$
$y = 8(x^2 + 2x) - 13$
$y = 8(x^2 + 2x + 4 - 4) - 13$
$y = 8[(x^2 + 2x + 4) - 4] - 13$
$y = 8[(x + 2)^2 - 4] - 13$
$y = 8(x + 2)^2 - 32 - 13$
$y = 8(x + 2)^2 - 45$

d) $f(x) = -3x^2 - 6x$
$f(x) = -3(x^2 - 6x - 9 + 9)$
$f(x) = -3[(x^2 - 6x - 9) + 9]$
$f(x) = -3[(x - 3)^2 + 9]$
$f(x) = -3(x - 3)^2 + 27$

13. The managers of a business are examining costs. It is more cost-effective for them to produce more items. However, if too many items are produced, their costs will rise because of factors such as storage and overstock. Suppose that they model the cost, C, of producing n thousand items with the function $C(n) = 75n^2 - 1800n + 60\,000$. Determine the number of items produced that will minimize their costs.

14. A gymnast is jumping on a trampoline. His height, h, in metres, above the floor on each jump is roughly approximated by the function $h(t) = -5t^2 + 10t + 4$, where t represents the time, in seconds, since he left the trampoline. Determine algebraically his maximum height on each jump.

15. Sandra is practising at an archery club. The height, h, in feet, of the arrow on one of her shots can be modelled as a function of time, t, in seconds, since it was fired using the function $h(t) = -16t^2 + 10t + 4$.

a) What is the maximum height of the arrow, in feet, and when does it reach that height?

b) Verify your solution in two different ways.

Did You Know?

The use of the bow and arrow dates back before recorded history and appears to have connections with most cultures worldwide. Archaeologists can learn great deal about the history of the ancestors of today's First Nations and Inuit populations in Canada through the study of various forms of spearheads and arrowheads, also referred to as *projectile points*.

16. Austin and Yuri were asked to convert the function $y = -6x^2 + 72x - 20$ to vertex form. Their solutions are shown.

Austin's solution:
$y = -6x^2 + 72x - 20$
$y = -6(x^2 + 12x) - 20$
$y = -6(x^2 + 12x + 36 - 36) - 20$
$y = -6[(x^2 + 12x + 36) - 36] - 20$
$y = -6[(x + 6) - 36] - 20$
$y = -6(x + 6) + 216 - 20$
$y = -6(x + 6) + 196$

Step 5 Changing the values of a, b, and c affects the position of the vertex, the steepness of the graph, and whether the graph opens upward ($a > 0$) or downward ($a < 0$). a affects the steepness and determines the direction of opening. b and a affect the value of the axis of symmetry, with b having a greater effect. c determines the value of the y-intercept.

3.3 Completing the Square, pages 192 to 197

1. a) $x^2 + 6x + 9$; $(x + 3)^2$
b) $x^2 - 4x + 4$; $(x - 2)^2$
c) $x^2 + 14x + 49$; $(x + 7)^2$
d) $x^2 - 2x + 1$; $(x - 1)^2$

2. a) $y = (x + 4)^2 - 16$; $(-4, -16)$
b) $y = (x - 9)^2 - 140$; $(9, -140)$
c) $y = (x - 5)^2 + 6$; $(5, 6)$
d) $y = (x + 16)^2 - 376$; $(-16, -376)$

3. a) $y = 2(x - 3)^2 - 18$; working backward, $y = 2(x - 3)^2 - 18$ results in the original function, $y = 2x^2 - 12x$.
b) $y = 6(x + 2)^2 - 7$; working backward, $y = 6(x + 2)^2 - 7$ results in the original function, $y = 6x^2 + 24x + 17$.
c) $y = 10(x - 8)^2 - 560$; working backward, $y = 10(x - 8)^2 - 560$ results in the original function, $y = 10x^2 - 160x + 80$.
d) $y = 3(x + 7)^2 - 243$; working backward, $y = 3(x + 7)^2 - 243$ results in the original function, $y = 3x^2 + 42x - 96$.

4. a) $f(x) = -4(x - 2)^2 + 16$; working backward, $f(x) = -4(x - 2)^2 + 16$ results in the original function, $f(x) = -4x^2 + 16x$.
b) $f(x) = -20(x + 10)^2 + 1757$; working backward, $f(x) = -20(x + 10)^2 + 1757$ results in the original function, $f(x) = -20x^2 - 400x - 243$.
c) $f(x) = -(x + 21)^2 + 941$; working backward, $f(x) = -(x + 21)^2 + 941$ results in the original function, $f(x) = -x^2 - 42x + 500$.
d) $f(x) = -7(x - 13)^2 + 1113$; working backward, $f(x) = -7(x - 13)^2 + 1113$ results in the original function, $f(x) = -7x^2 + 182x - 70$.

5. Verify each part by expanding the vertex form of the function and comparing with the standard form and by graphing both forms of the function.

6. a) minimum value of -11 when $x = -3$
b) minimum value of -11 when $x = 2$
c) maximum value of 25 when $x = -5$
d) maximum value of 5 when $x = 2$

7. a) minimum value of $-\frac{13}{4}$
b) minimum value of $\frac{1}{2}$
c) maximum value of 47
d) minimum value of -1.92
e) maximum value of 18.95
f) maximum value of 1.205

8. a) $y = \left(x + \frac{3}{4}\right)^2 - \frac{121}{16}$
b) $y = -\left(x + \frac{3}{16}\right)^2 + \frac{9}{256}$
c) $y = 2\left(x - \frac{5}{24}\right)^2 + \frac{263}{288}$

9. a) $f(x) = -2(x - 3)^2 + 8$
b) Example: The vertex of the graph is $(3, 8)$. From the function $f(x) = -2(x - 3)^2 + 8$, $p = 3$ and $q = 8$. So, the vertex is $(3, 8)$.

10. a) maximum value of 62; domain: $\{x \mid x \in \mathrm{R}\}$, range: $\{y \mid y \leq 62, y \in \mathrm{R}\}$
b) Example: By changing the function to vertex form, it is possible to find the maximum value since the function opens down and $p = 62$. This also helps to determine the range of the function. The domain is all real numbers for non-restricted quadratic functions.

11. Example: By changing the function to vertex form, the vertex is $\left(\frac{13}{4}, -\frac{3}{4}\right)$ or $(3.25, -0.75)$.

12. a) There is an error in the second line of the solution. You need to add and subtract the square of half the coefficient of the x-term.
$y = x^2 + 8x + 30$
$y = (x^2 + 8x + 16 - 16) + 30$
$y = (x + 4)^2 + 14$
b) There is an error in the second line of the solution. You need to add and subtract the square of half the coefficient of the x-term. There is also an error in the last line. The factor of 2 disappeared.
$f(x) = 2x^2 - 9x - 55$
$f(x) = 2[x^2 - 4.5x + 5.0625 - 5.0625] - 55$
$f(x) = 2[(x^2 - 4.5x + 5.0625) - 5.0625] - 55$
$f(x) = 2[(x - 2.25)^2 - 5.0625] - 55$
$f(x) = 2(x - 2.25)^2 - 10.125 - 55$
$f(x) = 2(x - 2.25)^2 - 65.125$
c) There is an error in the third line of the solution. You need to add and subtract the square of half the coefficient of the x-term.
$y = 8x^2 + 16x - 13$
$y = 8[x^2 + 2x] - 13$
$y = 8[x^2 + 2x + 1 - 1] - 13$
$y = 8[(x^2 + 2x + 1) - 1] - 13$
$y = 8[(x + 1)^2 - 1] - 13$
$y = 8(x + 1)^2 - 8 - 13$
$y = 8(x + 1)^2 - 21$

d) There are two errors in the second line of the solution. You need to factor the leading coefficient from the first two terms and add and subtract the square of half the coefficient of the x-term. There is also an error in the last line. The -3 factor was not distributed correctly.
$f(x) = -3x^2 - 6x$
$f(x) = -3[x^2 + 2x + 1 - 1]$
$f(x) = -3[(x^2 + 2x + 1) - 1]$
$f(x) = -3[(x + 1)^2 - 1]$
$f(x) = -3(x + 1)^2 + 3$

13. 12 000 items

14. 9 m

15. a) 5.56 ft; 0.31 s after being shot
b) Example: Verify by graphing and finding the vertex or by changing the function to vertex form and using the values of p and q to find the maximum value and when it occurs.

16. a) Austin got $+12x$ when dividing $72x$ by -6 and should have gotten $-12x$. He also forgot to square the quantity $(x + 6)$. Otherwise his work was correct and his answer should be $y = -6(x - 6)^2 + 196$. Yuri got an answer of -216 when he multiplied -6 by -36. He should have gotten 216 to get the correct answer of $y = -6(x - 6)^2 + 196$.
b) Example: To verify an answer, either work backward to show the functions are equivalent or use technology to show the graphs of the functions are identical.

17. 18 cm

18. a) The maximum revenue is \$151 250 when the ticket price is \$55.
b) 2750 tickets
c) Example: Assume that the decrease in ticket prices determines the same increase in ticket sales as indicated by the survey.

19. a) $R(n) = -50n^2 + 1000n + 100\,800$, where R is the revenue of the sales and n is the number of \$10 increases in price.
b) The maximum revenue is \$105 800 when the bikes are sold for \$460.
c) Example: Assume that the predictions of a decrease in sales for every increase in price holds true.

20. a) $P(n) = -0.1n^2 + n + 120$, where P is the production of peas, in kilograms, and n is the increase in plant rows.
b) The maximum production is 122.5 kg of peas when the farmer plants 35 rows of peas.
c) Example: Assume that the prediction holds true.

21. a) Answers may vary.
b) $A = -2w^2 + 90w$, where A is the area and w is the width.
c) 1012.5 m^2
d) Example: Verify the solution by graphing or changing the function to vertex form, where the vertex is (22.5, 1012.5).
e) Example: Assume that the measurements can be any real number.

22. The dimensions of the large field are 75 m by 150 m, and the dimensions of the small fields are 75 m by 50 m.

23. a) The two numbers are 14.5 and 14.5, and the maximum product is 210.25.
b) The two numbers are 6.5 and -6.5, and the minimum product is -42.25.

24. 8437.5 cm^2

25. $f(x) = -\frac{3}{4}\left(x - \frac{3}{4}\right)^2 + \frac{47}{64}$

26. a)
$y = ax^2 + bx + c$
$y = a\left(x^2 + \frac{b}{a}x\right) + c$
$y = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2}\right)\right) + c$
$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{ab^2}{4a^2} + c$
$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4a^2c - ab^2}{4a^2}$
$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{a(4ac - b^2)}{4a^2}$
$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$
b) $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$
c) Example: This formula can be used to find the vertex of any quadratic function without using an algebraic method to change the function to vertex form.

27. a) (3, 4)
b) $f(x) = 2(x - 3)^2 + 4$, so the vertex is (3, 4).
c) $a = a$, $p = -\frac{b}{2a}$, and $q = \frac{4ac - b^2}{4a}$

28. a) $A = -\left(\frac{4 + \pi}{8}\right)w^2 + 3w$
b) maximum area of $\frac{18}{4 + \pi}$, or approximately 2.52 m^2, when the width is $\frac{12}{4 + \pi}$, or approximately 1.68 m
c) Verify by graphing and comparing the vertex values, $\left(\frac{12}{4 + \pi}, \frac{18}{4 + \pi}\right)$, or approximately (1.68, 2.52).
d) width: $\frac{12}{4 + \pi}$ or approximately 1.68 m, length: $\frac{6}{4 + \pi}$ or approximately 0.84 m, radius: $\frac{6}{4 + \pi}$ or approximately 0.84 m; Answers may vary.