Key Ideas

- A trigonometric identity is an equation involving trigonometric functions that is true for all permissible values of the variable.
- You can verify trigonometric identities
  - numerically by substituting specific values for the variable
  - graphically, using technology
- Verifying that two sides of an equation are equal for given values, or that they appear equal when graphed, is not sufficient to conclude that the equation is an identity.
- You can use trigonometric identities to simplify more complicated trigonometric expressions.
- The reciprocal identities are

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

- The quotient identities are

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

- The Pythagorean identities are

$$\cos^2 x + \sin^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad \cot^2 x + 1 = \csc^2 x$$

Check Your Understanding

Practise

1. Determine the non-permissible values of x, in radians, for each expression.

a) $\dfrac{\cos x}{\sin x}$ **b)** $\dfrac{\sin x}{\tan x}$

c) $\dfrac{\cot x}{1 - \sin x}$ **d)** $\dfrac{\tan x}{\cos x + 1}$

2. Why do some identities have non-permissible values?

3. Simplify each expression to one of the three primary trigonometric functions, $\sin x$, $\cos x$ or $\tan x$. For part a), verify graphically, using technology, that the given expression is equivalent to its simplified form.

a) $\sec x \sin x$

b) $\sec x \cot x \sin^2 x$

c) $\dfrac{\cos x}{\cot x}$

4. Simplify, and then rewrite each expression as one of the three reciprocal trigonometric functions, $\csc x$, $\sec x$, or $\cot x$.

a) $\left(\dfrac{\cos x}{\tan x}\right)\left(\dfrac{\tan x}{\sin x}\right)$

b) $\csc x \cot x \sec x \sin x$

c) $\dfrac{\cos x}{1 - \sin^2 x}$

5. a) Verify that the equation $\dfrac{\sec x}{\tan x + \cot x} = \sin x$ is true for $x = 30°$ and for $x = \dfrac{\pi}{4}$.

b) What are the non-permissible values of the equation in the domain $0° \leq x < 360°$?

6. Consider the equation
$$\frac{\sin x \cos x}{1 + \cos x} = \frac{1 - \cos x}{\tan x}.$$

a) What are the non-permissible values, in radians, for this equation?

b) Graph the two sides of the equation using technology, over the domain $0 \le x < 2\pi$. Could it be an identity?

c) Verify that the equation is true when $x = \frac{\pi}{4}$. Use exact values for each expression in the equation.

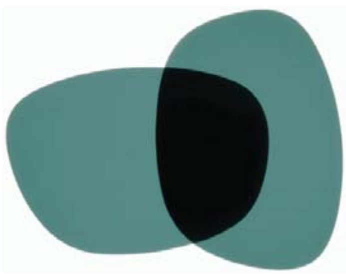
Apply

7. When a polarizing lens is rotated through an angle θ over a second lens, the amount of light passing through both lenses decreases by $1 - \sin^2 \theta$.

a) Determine an equivalent expression for this decrease using only cosine.

b) What fraction of light is lost when $\theta = \frac{\pi}{6}$?

c) What percent of light is lost when $\theta = 60°$?



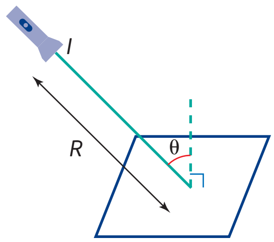
8. Compare $y = \sin x$ and $y = \sqrt{1 - \cos^2 x}$ by completing the following.

a) Verify that $\sin x = \sqrt{1 - \cos^2 x}$ for $x = \frac{\pi}{3}$, $x = \frac{5\pi}{6}$, and $x = \pi$.

b) Graph $y = \sin x$ and $y = \sqrt{1 - \cos^2 x}$ in the same window.

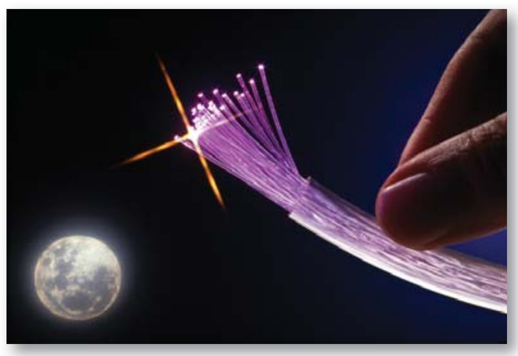
c) Determine whether $\sin x = \sqrt{1 - \cos^2 x}$ is an identity. Explain your answer.

9. Illuminance (E) is a measure of the amount of light coming from a light source and falling onto a surface. If the light is projected onto the surface at an angle θ, measured from the perpendicular, then a formula relating these values is $\sec \theta = \frac{I}{ER^2}$, where I is a measure of the luminous intensity and R is the distance between the light source and the surface.



a) Rewrite the formula so that E is isolated and written in terms of $\cos \theta$.

b) Show that $E = \frac{I \cot \theta}{R^2 \csc \theta}$ is equivalent to your equation from part a).



Fibre optic cable

10. Simplify $\frac{\csc x}{\tan x + \cot x}$ to one of the three primary trigonometric ratios. What are the non-permissible values of the original expression in the domain $0 \le x < 2\pi$?

11. a) Determine graphically, using technology, whether the expression $\frac{\csc^2 x - \cot^2 x}{\cos x}$ appears to be equivalent to csc *x* or sec *x*.

b) What are the non-permissible values, in radians, for the identity from part a)?

c) Express $\frac{\csc^2 x - \cot^2 x}{\cos x}$ as the single reciprocal trigonometric ratio that you identified in part a).

12. a) Substitute $x = \frac{\pi}{4}$ into the equation $\frac{\cot x}{\sec x} + \sin x = \csc x$ to determine whether it could be an identity. Use exact values.

b) Algebraically confirm that the expression on the left side simplifies to csc *x*.

13. Stan, Lina, and Giselle are working together to try to determine whether the equation $\sin x + \cos x = \tan x + 1$ is an identity.

a) Stan substitutes $x = 0$ into each side of the equation. What is the result?

b) Lina substitutes $x = \frac{\pi}{2}$ into each side of the equation. What does she observe?

c) Stan points out that Lina's choice is not permissible for this equation. Explain why.

d) Giselle substitutes $x = \frac{\pi}{4}$ into each side of the equation. What does she find?

e) Do the three students have enough information to conclude whether or not the given equation is an identity? Explain.

14. Simplify $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$.

Extend

15. Given $\csc^2 x + \sin^2 x = 7.89$, find the value of $\frac{1}{\csc^2 x} + \frac{1}{\sin^2 x}$.

16. Show algebraically that

$$\frac{1}{1 + \sin \theta} + \frac{1}{1 - \sin \theta} = 2 \sec^2 \theta$$

is an identity.

17. Determine an expression for *m* that makes $\frac{2 - \cos^2 x}{\sin x} = m + \sin x$ an identity.

Create Connections

C1 Explain how a student who does not know the $\cot^2 x + 1 = \csc^2 x$ form of the Pythagorean identity could simplify an expression that contained the expression $\cot^2 x + 1$ using the fact that $1 = \frac{\sin^2 x}{\sin^2 x}$.

C2 For some trigonometric expressions, multiplying by a conjugate helps to simplify the expression. Simplify $\frac{\sin \theta}{1 + \cos \theta}$ by multiplying the numerator and the denominator by the conjugate of the denominator, $1 - \cos \theta$. Describe how this process helps to simplify the expression.

C3 MINI LAB Explore the effect of different domains on apparent identities.

Materials
- graphing calculator

Step 1 Graph the two functions $y = \tan x$ and $y = \left|\frac{\sin x}{\cos x}\right|$ on the same grid, using a domain of $0 \le x < \frac{\pi}{2}$. Is there graphical evidence that $\tan x = \left|\frac{\sin x}{\cos x}\right|$ is an identity? Explain.

Step 2 Graph the two functions $y = \tan x$ and $y = \left|\frac{\sin x}{\cos x}\right|$ again, using the expanded domain $-2\pi < x \le 2\pi$. Is the equation $\tan x = \left|\frac{\sin x}{\cos x}\right|$ an identity? Explain.

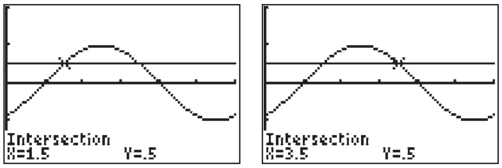
Step 3 Find and record a different trigonometric equation that is true over a restricted domain but is not an identity when all permissible values are checked. Compare your answer with that of a classmate.

Step 4 How does this activity show the weakness of using graphical and numerical methods for verifying potential identities?

11. Amplitude is 120; period is 0.0025 s or 2.5 ms.

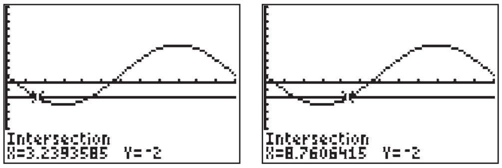
12. The minimum depth of 2 m occurs at 0 h, 12 h, and 24 hour. The maximum depth of 8 m occurs at 6 h and 18 h.

13. a)



$x = 1.5 + 6n$ radians and $x = 3.5 + 6n$ radians, where n is an integer

b)



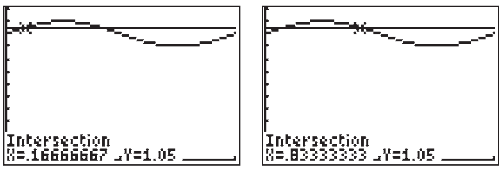
$x \approx 3.24° + (24°)n$ and

$x \approx 8.76° + (24°)n$, where n is an integer

14. Example: Graph II has half the period of graph I. Graph I represents a cosine curve with no phase shift. Graph II represents a sine curve with no phase shift. Graph I and II have the same amplitude and both graphs have no vertical translations.

15. a) $h = 0.1 \sin \pi t + 1$, where t represents the time, in seconds, and h represents the height of the mass, in metres, above the floor

b)



approximately 0.17 s and 0.83 s

c) $t = \frac{1}{6}$ or 0.1666… and $t = \frac{5}{6}$ or 0.8333

16. a) $y = 3 \sin 2\left(x - \frac{\pi}{4}\right) - 1$ b) $y = -3 \cos 2x - 1$

17. a) A, B b) A, B or C, D, E c) B

Chapter 6 Trigonometric Identities

6.1 Reciprocal, Quotient, and Pythagorean Identities, pages 296 to 298

1. a) $x \neq \pi n;\ n \in I$ b) $x \neq \left(\frac{\pi}{2}\right)n,\ n \in I$

c) $x \neq \frac{\pi}{2} + 2\pi n$ and $x \neq \pi n,\ n \in I$

d) $x \neq \frac{\pi}{2} + \pi n$ and $x \neq \pi + 2\pi n,\ n \in I$

2. Some identities will have non-permissible values because they involve trigonometric functions that have non-permissible values themselves or a function occurs in a denominator. For example, an identity involving $\sec \theta$ has non-permissible values $\theta \neq 90° + 180°n$, where $n \in I$, because these are the non-permissible values for the function.

3. a) $\tan x$ b) $\sin x$ c) $\sin x$

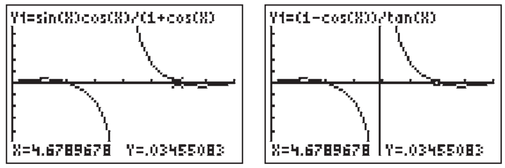
4. a) $\cot x$ b) $\csc x$ c) $\sec x$

5. a) When substituted, both values satisfied the equation.

b) $x \neq 0°, 90°, 180°, 270°$

6. a) $x \neq \pi + 2\pi n,\ n \in I;\ x \neq \frac{\pi}{2} + \pi n,\ n \in I$

b)



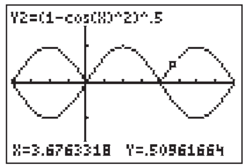
Yes, it appears to be an identity.

c) The equation is verified for $x = \frac{\pi}{4}$.

7. a) $\cos^2 \theta$ b) 0.75 c) 25%

8. a) All three values check when substituted.

b)



c) The equation is not an identity since taking the square then the square root removes the negative sign and $\sin x$ is negative from π to 2π.

9. a) $E = \frac{I \cos \theta}{R^2}$ b) $E = \frac{I \cot \theta}{R^2 \csc \theta}$

$$E = \frac{I\left(\frac{\cos \theta}{\sin \theta}\right)}{R^2\left(\frac{1}{\sin \theta}\right)}$$

$$E = \left(\frac{I \cos \theta}{\sin \theta}\right)\left(\frac{\sin \theta}{R^2}\right)$$

$$E = \frac{I \cos \theta}{R^2}$$

10. $\cos x,\ x \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

11. a) It appears to be equivalent to $\sec x$.

b) $x \neq \frac{\pi}{2} + \pi n,\ n \in I$

c)
$$\frac{\csc^2 x - \cot^2 x}{\cos x} = \frac{\frac{1}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x}}{\cos x}$$
$$= \frac{\frac{1 - \cos^2 x}{\sin^2 x}}{\cos x}$$
$$= \frac{\frac{\sin^2 x}{\sin^2 x}}{\cos x}$$
$$= \frac{1}{\cos x}$$
$$= \sec x$$

12. a) Yes, it could be an identity.

b)
$$\frac{\cot x}{\sec x} + \sin x = \frac{\cos x}{\sin x} \div \frac{1}{\cos x} + \sin x$$
$$= \frac{\cos^2 x}{\sin x} + \sin x$$
$$= \frac{\cos^2 x + \sin^2 x}{\sin x}$$
$$= \csc x$$

13. a) $1 = 1$

b) The left side = 1, but the right side is undefined.

c) The chosen value is not permissible for the $\tan x$ function.

d) The left side $= \frac{2}{\sqrt{2}}$, but the right side = 2.

e) Giselle has found a permissible value for which the equation is not true, so they can conclude that it is not an identity.

14. 2

15. 7.89

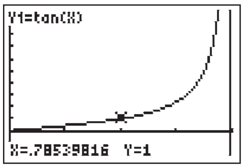
16. $\frac{1}{1 + \sin\theta} + \frac{1}{1 - \sin\theta} = \frac{1 - \sin\theta + 1 + \sin\theta}{(1 - \sin\theta)(1 + \sin\theta)}$
$= \frac{2}{(1 - \sin^2\theta)}$
$= 2\sec^2\theta$

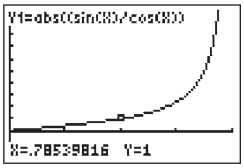
17. $m = \csc x$

C1 $\cot^2 x + 1$
$= \frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x}$
$= \frac{\cos^2 x + \sin^2 x}{\sin^2 x}$
$= \frac{1}{\sin^2 x}$
$= \csc^2 x$

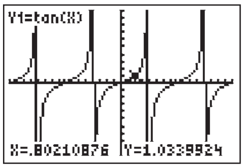
C2 $\left(\frac{\sin\theta}{1 + \cos\theta}\right)\left(\frac{1 - \cos\theta}{1 - \cos\theta}\right)$
$= \frac{\sin\theta - \sin\theta\cos\theta}{1 - \cos^2\theta}$
$= \frac{\sin\theta - \sin\theta\cos\theta}{\sin^2\theta}$
$= \frac{1 - \cos\theta}{\sin\theta}$
It helps to simplify by creating an opportunity to use the Pythagorean identity.

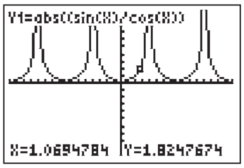
C3 **Step 1**





Yes, over this domain it is an identity.
Step 2





The equation is not an identity since the graphs of the two sides are not the same.
Step 3 Example: $y = \cot\theta$ and $y = \left|\frac{\cos\theta}{\sin\theta}\right|$ are identities over the domain $0 \le \theta \le \frac{\pi}{2}$ but not over the domain $-2\pi \le \theta \le 2\pi$
Step 4 The weakness with this approach is that for some more complicated identities you may think it is an identity when really it is only an identity over that domain.

6.2 Sum, Difference, and Double-angle Identities, pages 306 to 308

1. a) $\cos 70°$ **b)** $\sin 35°$ **c)** $\cos 38°$
d) $\sin\frac{\pi}{4}$ **e)** $4\sin\frac{2\pi}{3}$

2. a) $\cos 60° = 0.5$ **b)** $\sin 45° = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$
c) $\cos\frac{\pi}{3} = 0.5$ **d)** $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

3. $\cos 2x = 1 - 2\sin^2 x$;
$1 - \cos 2x = 1 - 1 + 2\sin^2 x = 2\sin^2 x$

4. a) $\sin\frac{\pi}{2}$ **b)** $6\sin 48°$ **c)** $\tan 152°$ **d)** $\cos\frac{\pi}{3}$
e) $-\cos\frac{\pi}{6}$

5. a) $\sin\theta$ **b)** $\cos x$ **c)** $\cos\theta$ **d)** $\cos x$

6. Example: When $x = 60°$ and $y = 30°$, then left side $= 0.5$, but right side ≈ 0.366.

7. $\cos(90° - x) = \cos 90°\cos x + \sin 90°\sin x$
$= \sin x$

8. a) $\frac{\sqrt{3} - 1}{2\sqrt{2}}$ or $\frac{\sqrt{6} - \sqrt{2}}{4}$ **b)** $\frac{-\sqrt{3} + 1}{\sqrt{3} + 1}$ or $\sqrt{3} - 2$
c) $\frac{1 + \sqrt{3}}{2\sqrt{2}}$ or $\frac{\sqrt{2} + \sqrt{6}}{4}$ **d)** $\frac{-\sqrt{3} - 1}{2\sqrt{2}}$ or $\frac{-\sqrt{6} - \sqrt{2}}{4}$
e) $\sqrt{2}(1 + \sqrt{3})$ **f)** $\frac{1 - \sqrt{3}}{2\sqrt{2}}$ or $\frac{\sqrt{2} - \sqrt{6}}{4}$

9. a) $P = 1000\sin(x + 113.5°)$
b) **i)** 101.056 W/m² **ii)** 310.676 W/m²
iii) −50.593 W/m²
c) The answer in part iii) is negative which means that there is no sunlight reaching Igloolik. At latitude 66.5°, the power received is 0 W/m².

10. $-2\cos x$

11. a) $\frac{119}{169}$ **b)** $-\frac{120}{169}$ **c)** $-\frac{12}{13}$

12. a) Both sides are equal for this value.
b) Both sides are equal for this value.
c) $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$
$= \frac{2\tan x}{1 - \tan^2 x}\left(\frac{\cos^2 x}{\cos^2 x}\right)$
$= \frac{2\left(\frac{\sin x}{\cos x}\right)(\cos^2 x)}{\left(1 - \frac{\sin^2 x}{\cos^2 x}\right)\cos^2 x}$
$= \frac{2\sin x\cos x}{\cos^2 x - \sin^2 x}$

13. a) $d = \frac{v_o^2\sin 2\theta}{g}$ **b)** 45°
c) It is easier after applying the double-angle identity since there is only one trigonometric function whose value has to be found.

14. $k - 1$

15. a) $\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$
$= \cos^2 x - \sin^2 x$
$= \cos 2x$
b) $\frac{\csc^2 x - 2}{\csc^2 x} = 1 - \frac{2}{\csc^2 x}$
$= 1 - 2\sin^2 x$
$= \cos 2x$

16. a) $\frac{1 - \cos 2x}{2} = \frac{1 - 1 + 2\sin^2 x}{2} = \sin^2 x$
b) $\frac{4 - 8\sin^2 x}{2\sin x\cos x} = \frac{4\cos 2x}{\sin 2x} = \frac{4}{\tan 2x}$

17. $-\frac{2}{\sqrt{29}}$

18. $k = 3$

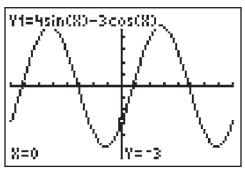
19. a) $0.9928, -0.392\ 82$ or $\frac{\pm 4\sqrt{3} + 3}{10}$
b) 0.9500 or $\frac{\sqrt{5} + 2\sqrt{3}}{6}$

20. a) $\frac{56}{65}$ **b)** $\frac{63}{65}$ **c)** $\frac{-7}{25}$ **d)** $\frac{24}{25}$

21. a) $\sin x$ **b)** $\tan x$

22. $\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1$
$\frac{\cos x + 1}{2} = \cos^2\left(\frac{x}{2}\right)$
$\pm\sqrt{\frac{\cos x + 1}{2}} = \cos\frac{x}{2}$

23. a)



b) $a = 5$, $c = 37°$
c) $y = 5\sin(x - 36.87°)$