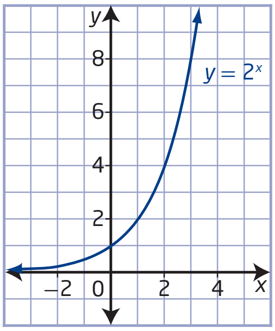
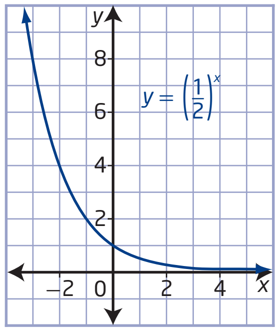
Key Ideas

- An exponential function of the form $y = c^x$, $c > 0$,
  - is increasing for $c > 1$
  - is decreasing for $0 < c < 1$
  - is neither increasing nor decreasing for $c = 1$
  - has a domain of $\{x \mid x \in \mathrm{R}\}$
  - has a range of $\{y \mid y > 0, y \in \mathrm{R}\}$
  - has a y-intercept of 1
  - has no x-intercept
  - has a horizontal asymptote at $y = 0$





Check Your Understanding

Practise

1. Decide whether each of the following functions is exponential. Explain how you can tell.

a) $y = x^3$

b) $y = 6^x$

c) $y = x^{\frac{1}{2}}$

d) $y = 0.75^x$

2. Consider the following exponential functions:

- $f(x) = 4^x$
- $g(x) = \left(\frac{1}{4}\right)^x$
- $h(x) = 2^x$

a) Which is greatest when $x = 5$?

b) Which is greatest when $x = -5$?

c) For which value of x do all three functions have the same value? What is this value?

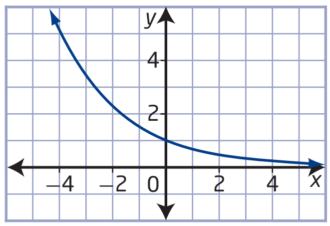
3. Match each exponential function to its corresponding graph.

a) $y = 5^x$

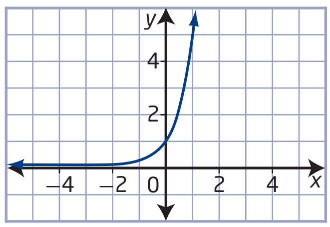
b) $y = \left(\frac{1}{4}\right)^x$

c) $y = \left(\frac{2}{3}\right)^x$

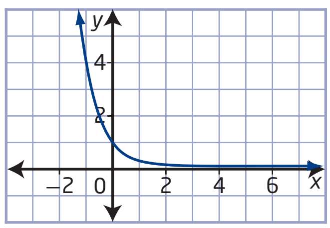
A



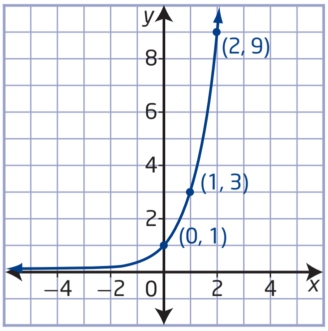
B

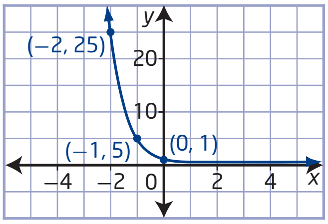


C



4. Write the function equation for each graph of an exponential function.

a) 

b) 

5. Sketch the graph of each exponential function. Identify the domain and range, the y-intercept, whether the function is increasing or decreasing, and the equation of the horizontal asymptote.

a) $g(x) = 6^x$

b) $h(x) = 3.2^x$

c) $f(x) = \left(\frac{1}{10}\right)^x$

d) $k(x) = \left(\frac{3}{4}\right)^x$

Apply

6. Each of the following situations can be modelled using an exponential function. Indicate which situations require a value of $c > 1$ (growth) and which require a value of $0 < c < 1$ (decay). Explain your choices.

a) Bacteria in a Petri dish double their number every hour.

b) The half-life of the radioactive isotope actinium-225 is 10 days.

c) As light passes through every 1-m depth of water in a pond, the amount of light available decreases by 20%.

d) The population of an insect colony triples every day.

7. A flu virus is spreading through the student population of a school according to the function $N = 2^t$, where N is the number of people infected and t is the time, in days.

a) Graph the function. Explain why the function is exponential.

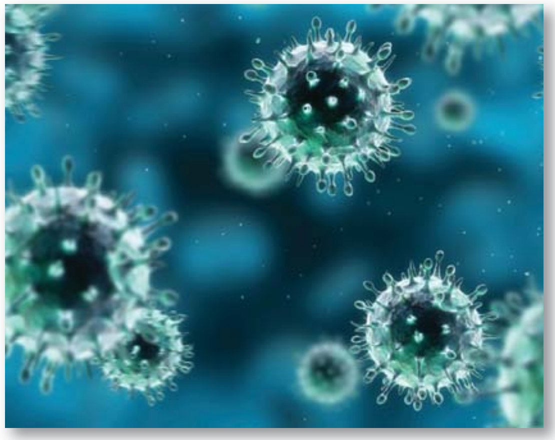
b) How many people have the virus at each time?

i) at the start when $t = 0$

ii) after 1 day

iii) after 4 days

iv) after 10 days



8. If a given population has a constant growth rate over time and is never limited by food or disease, it exhibits exponential growth. In this situation, the growth rate alone controls how quickly (or slowly) the population grows. If a population, P, of fish, in hundreds, experiences exponential growth at a rate of 10% per year, it can be modelled by the exponential function $P(t) = 1.1^t$, where t is time, in years.

a) Why is the base for the exponential function that models this situation 1.1?

b) Graph the function $P(t) = 1.1^t$. What are the domain and range of the function?

c) If the same population of fish decreased at a rate of 5% per year, how would the base of the exponential model change?

d) Graph the new function from part c). What are the domain and range of this function?

9. Scuba divers know that the deeper they dive, the more light is absorbed by the water above them. On a dive, Petra's light meter shows that the amount of light available decreases by 10% for every 10 m that she descends.

a) Write the exponential function that relates the amount, L, as a percent expressed as a decimal, of light available to the depth, d, in 10-m increments.

b) Graph the function.

c) What are the domain and range of the function for this situation?

d) What percent of light will reach Petra if she dives to a depth of 25 m?

10. The CANDU (CANada Deuterium Uranium) reactor is a Canadian-invented pressurized heavy-water reactor that uses uranium-235 (U-235) fuel with a half-life of approximately 700 million years.

a) What exponential function can be used to represent the radioactive decay of 1 kg of U-235? Define the variables you use.

b) Graph the function.

c) How long will it take for 1 kg of U-235 to decay to 0.125 kg?

d) Will the sample in part c) decay to 0 kg? Explain.

Did You Know?

Canada is one of the world's leading uranium producers, accounting for 18% of world primary production. All of the uranium produced in Canada comes from Saskatchewan mines. The energy potential of Saskatchewan's uranium reserves is approximately equivalent to 4.5 billion tonnes of coal or 17.5 billion barrels of oil.

11. Money in a savings account earns compound interest at a rate of 1.75% per year. The amount, A, of money in an account can be modelled by the exponential function $A = P(1.0175)^n$, where P is the amount of money first deposited into the savings account and n is the number of years the money remains in the account.

a) Graph this function using a value of $P = \$1$ as the initial deposit.

b) Approximately how long will it take for the deposit to triple in value?

c) Does the amount of time it takes for a deposit to triple depend on the value of the initial deposit? Explain.

d) In finance, the *rule of 72* is a method of estimating an investment's doubling time when interest is compounded annually. The number 72 is divided by the annual interest rate to obtain the approximate number of years required for doubling. Use your graph and the rule of 72 to approximate the doubling time for this investment.

12. Statistics indicate that the world population since 1995 has been growing at a rate of about 1.27% per year. United Nations records estimate that the world population in 2011 was approximately 7 billion. Assuming the same exponential growth rate, when will the population of the world be 9 billion?

Extend

13. a) On the same set of axes, sketch the graph of the function $y = 5^x$, and then sketch the graph of the inverse of the function by reflecting its graph in the line $y = x$.

b) How do the characteristics of the graph of the inverse of the function relate to the characteristics of the graph of the original exponential function?

c) Express the equation of the inverse of the exponential function in terms of y. That is, write $x = F(y)$.

b) $x = \frac{\pi}{2} + \pi n,\ n \in I,\ x = \frac{7\pi}{6} + 2\pi n,\ n \in I,$
$x = \frac{11\pi}{6} + 2\pi n,\ n \in I$

27. a) This is an identity so all θ are a solution.
b) Yes, because the left side can be simplified to 1.

Unit 2 Test, pages 328 to 329

1. B **2.** D **3.** C **4.** C **5.** B **6.** D **7.** C **8.** A

9. $-\frac{\sqrt{3}}{2}$

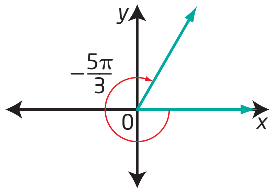
10. $-\frac{2}{3}, \frac{2}{3}$

11. $\frac{7}{13\sqrt{2}}$ or $\frac{7\sqrt{2}}{26}$

12. 1.5, 85.9°

13. $-\frac{11\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}$

14. a)

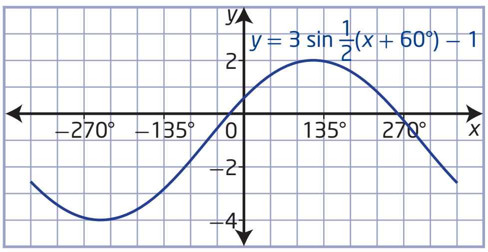


b) −300°
c) $-\frac{5\pi}{3} \pm 2\pi n,\ n \in N$
d) No, following the equation above it is impossible to obtain $\frac{10\pi}{3}$.

15. $x = 0.412, 2.730, 4.712$

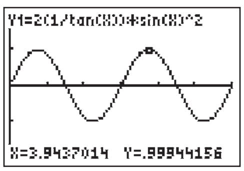
16. Sam is correct, there are four solutions in the given domain. Pat made an error when finding the square root. Pat forgot to solve for the positive and negative solutions.

17. a)



b) $-4 \le y \le 2$
c) amplitude 3, period 720°, phase shift 60° left, vertical displacement 1 unit down
d) $x \approx -21°, 261°$

18. a)



b) $g(\theta) = \sin 2\theta$
c) $f(\theta) = 2\cot\theta\sin^2\theta$
$= \frac{2\cos\theta\sin^2\theta}{\sin\theta}$
$= 2\cos\theta\sin\theta$
$= \sin 2\theta$
$= g(\theta)$

19. a) It is true: both sides have the same value.
b) $x \neq \frac{\pi n}{2},\ n \in I$
c) Left Side
$= \tan x + \frac{1}{\tan x}$
$= \frac{\tan^2 x + 1}{\tan x}$
$= \frac{\sec^2 x}{\tan x}$
$= \sec x\left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{\sin x}\right)$
$= \frac{\sec x}{\sin x}$
= Right Side

20. a) 6.838 m **b)** 12.37 h **c)** 3.017 m

Chapter 7 Exponential Functions

7.1 Characteristics of Exponential Functions, pages 342 to 345

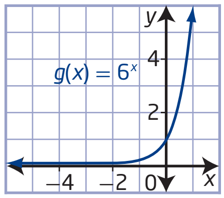
1. a) No, the variable is not the exponent.
b) Yes, the base is greater than 0 and the variable is the exponent.
c) No, the variable is not the exponent.
d) Yes, the base is greater than 0 and the variable is the exponent.

2. a) $f(x) = 4^x$ **b)** $g(x) = \left(\frac{1}{4}\right)^x$
c) $x = 0$, which is the y-intercept

3. a) B **b)** C **c)** A

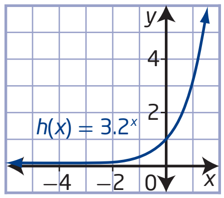
4. a) $f(x) = 3^x$ **b)** $f(x) = \left(\frac{1}{5}\right)^x$

5. a)



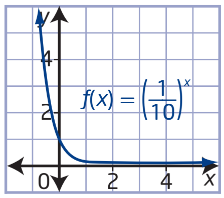
domain $\{x \mid x \in R\}$, range $\{y \mid y > 0, y \in R\}$, y-intercept 1, function increasing, horizontal asymptote $y = 0$

b)



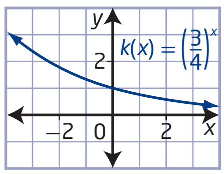
domain $\{x \mid x \in R\}$, range $\{y \mid y > 0, y \in R\}$, y-intercept 1, function increasing, horizontal asymptote $y = 0$

c)



domain $\{x \mid x \in R\}$, range $\{y \mid y > 0, y \in R\}$, y-intercept 1, function decreasing, horizontal asymptote $y = 0$

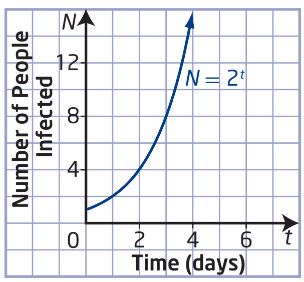
d)



domain $\{x \mid x \in R\}$, range $\{y \mid y > 0, y \in R\}$, y-intercept 1, function decreasing, horizontal asymptote $y = 0$

6. a) $c > 1$; number of bacteria increases over time
b) $0 < c < 1$; amount of actinium-225 decreases over time
c) $0 < c < 1$; amount of light decreases with depth
d) $c > 1$; number of insects increases over time

7. a)

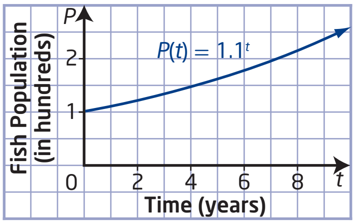


The function $N = 2^t$ is exponential since the base is greater than zero and the variable t is an exponent.

b) **i)** 1 person **ii)** 2 people
iii) 16 people **iv)** 1024 people

8. a) If the population increases by 10% each year, the population becomes 110% of the previous year's population. So, the growth rate is 110% or 1.1 written as a decimal.

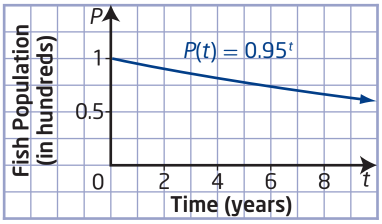
b)



domain $\{t \mid t \geq 0, t \in \mathrm{R}\}$ and range $\{P \mid P \geq 100, P \in \mathrm{R}\}$

c) The base of the exponent would become 100% − 5% or 95%, written as 0.95 in decimal form.

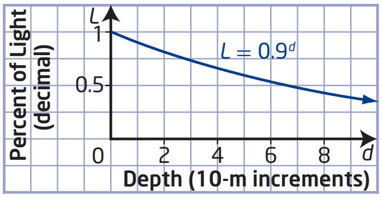
d)



domain $\{t \mid t \geq 0, t \in \mathrm{R}\}$ and range $\{P \mid 0 < P \leq 100, P \in \mathrm{R}\}$

9. a) $L = 0.9^d$

b)

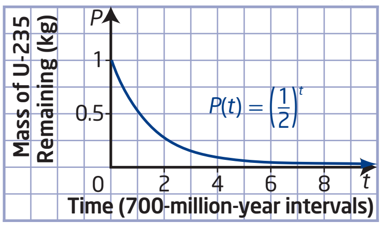


c) domain $\{d \mid d \geq 0, d \in \mathrm{R}\}$ and range $\{L \mid 0 < L \leq 1, L \in \mathrm{R}\}$

d) 76.8%

10. a) Let P represent the percent, as a decimal, of U-235 remaining. Let t represent time, in 700-million-year intervals. $P(t) = \left(\frac{1}{2}\right)^t$

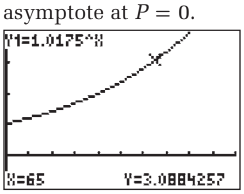
b)



c) 2.1×10^9 years

d) No, the sample of U-235 will never decay to 0 kg, since the graph of $P(t) = \left(\frac{1}{2}\right)^t$ has a horizontal asymptote at $P = 0$.

11. a)



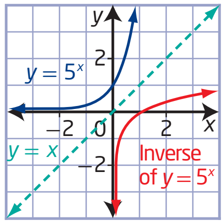
b) 64 years

c) No; since the amount invested triples, it does not matter what initial investment is made.

d) graph: 40 years; rule of 72: 41 years

12. 19.9 years

13. a)



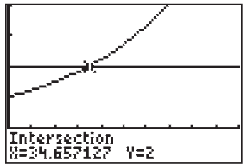
b) The x- and y-coordinates of any point and the domains and ranges are interchanged. The horizontal asymptote becomes a vertical asymptote.

c) $x = 5^y$

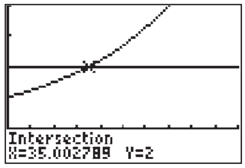
14. a) Another way to express $D = 2^{-\varphi}$ is as $D = \left(\frac{1}{2}\right)^{\varphi}$, which indicates a decreasing exponential function. Therefore, a negative value of φ represents a greater value of D.

b) The diameter of fine sand (0.125 mm) is $\frac{1}{256}$ the diameter of course gravel (32 mm).

15. a) 34.7 years

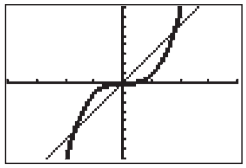


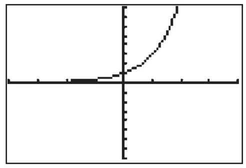
b) 35 years



c) The results are similar, but the continuous compounding function gives a shorter doubling period by approximately 0.3 years.

C1 a)





b)

Feature	$f(x) = 3x$	$g(x) = x^3$	$h(x) = 3^x$
domain	$\{x \mid x \in \mathrm{R}\}$	$\{x \mid x \in \mathrm{R}\}$	$\{x \mid x \in \mathrm{R}\}$
range	$\{y \mid y \in \mathrm{R}\}$	$\{y \mid y \in \mathrm{R}\}$	$\{y \mid y > 0, y \in \mathrm{R}\}$
intercepts	x-intercept 0, y-intercept 0	x-intercept 0, y-intercept 0	no x-intercept, y-intercept 1
equations of asymptotes	none	none	$y = 0$

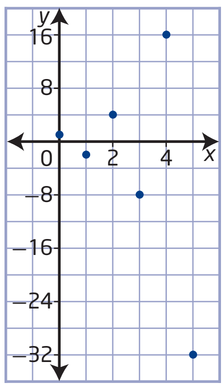
c) Example: All three functions have the same domain, and each of their graphs has a y-intercept. The functions $f(x)$ and $g(x)$ have all key features in common.

d) Example: The function $h(x)$ is the only function with an asymptote, which restricts its range and results in no x-intercept.

C2 a)

x	$f(x)$
0	1
1	−2
2	4
3	−8
4	16
5	−32

b)



c) No, the points do not form a smooth curve. The locations of the points alternate between above the x-axis and below the x-axis.