Key Ideas

- To sketch the graph of an exponential function of the form $y = a(c)^{b(x-h)} + k$, apply transformations to the graph of $y = c^x$, where $c > 0$. The transformations represented by a and b may be applied in any order before the transformations represented by h and k.
- The parameters a, b, h, and k in exponential functions of the form $y = a(c)^{b(x-h)} + k$ correspond to the following transformations:
  - a corresponds to a vertical stretch about the x-axis by a factor of $|a|$ and, if $a < 0$, a reflection in the x-axis.
  - b corresponds to a horizontal stretch about the y-axis by a factor of $\frac{1}{|b|}$ and, if $b < 0$, a reflection in the y-axis.
  - h corresponds to a horizontal translation left or right.
  - k corresponds to a vertical translation up or down.
- Transformed exponential functions can be used to model real-world applications of exponential growth or decay.

Check Your Understanding

Practise

1. Match each function with the corresponding transformation of $y = 3^x$.

a) $y = 2(3)^x$ **b)** $y = 3^{x-2}$

c) $y = 3^x + 4$ **d)** $y = 3^{\frac{x}{5}}$

A translation up

B horizontal stretch

C vertical stretch

D translation right

2. Match each function with the corresponding transformation of $y = \left(\frac{3}{5}\right)^x$.

a) $y = \left(\frac{3}{5}\right)^{x+1}$ **b)** $y = -\left(\frac{3}{5}\right)^x$

c) $y = \left(\frac{3}{5}\right)^{-x}$ **d)** $y = \left(\frac{3}{5}\right)^x - 2$

A reflection in the x-axis

B reflection in the y-axis

C translation down

D translation left

3. For each function, identify the parameters a, b, h, and k and the type of transformation that corresponds to each parameter.

a) $f(x) = 2(3)^x - 4$

b) $g(x) = 6^{x-2} + 3$

c) $m(x) = -4(3)^{x+5}$

d) $y = \left(\frac{1}{2}\right)^{3(x-1)}$

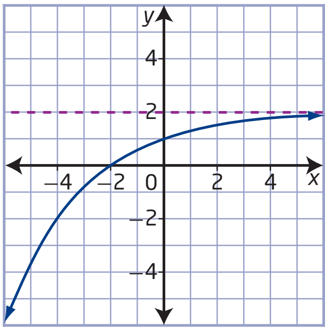
e) $n(x) = -\frac{1}{2}(5)^{2(x-4)} + 3$

f) $y = -\left(\frac{2}{3}\right)^{2x-2}$

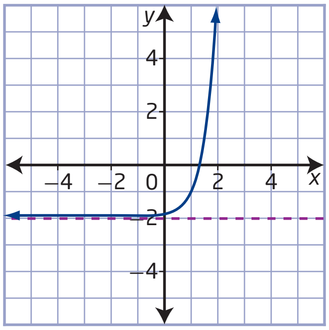
g) $y = 1.5(0.75)^{\frac{x-4}{2}} - \frac{5}{2}$

4. Without using technology, match each graph with the corresponding function. Justify your choice.

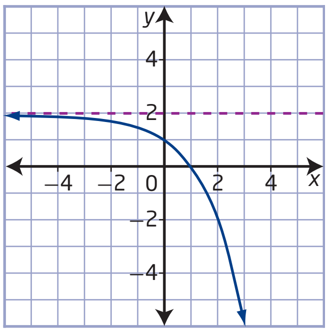
a)



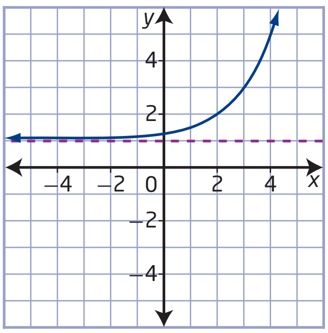
b)



c)



d)



A $y = 3^{2(x-1)} - 2$

B $y = 2^{x-2} + 1$

C $y = -\left(\frac{1}{2}\right)^{\frac{1}{2}x} + 2$

D $y = -\frac{1}{2}(4)^{\frac{1}{2}(x+1)} + 2$

5. The graph of $y = 4^x$ is transformed to obtain the graph of $y = \frac{1}{2}(4)^{-(x-3)} + 2$.

a) What are the parameters and corresponding transformations?

b) Copy and complete the table.

$y = 4^x$	$y = 4^{-x}$	$y = \frac{1}{2}(4)^{-x}$	$y = \frac{1}{2}(4)^{-(x-3)} + 2$
$\left(-2, \frac{1}{16}\right)$			
$\left(-1, \frac{1}{4}\right)$			
(0, 1)			
(1, 4)			
(2, 16)			

c) Sketch the graph of $y = \frac{1}{2}(4)^{-(x-3)} + 2$.

d) Identify the domain, range, equation of the horizontal asymptote, and any intercepts for the function $y = \frac{1}{2}(4)^{-(x-3)} + 2$.

6. For each function,

i) state the parameters a, b, h, and k

ii) describe the transformation that corresponds to each parameter

iii) sketch the graph of the function

iv) identify the domain, range, equation of the horizontal asymptote, and any intercepts

a) $y = 2(3)^x + 4$

b) $m(r) = -(2)^{r-3} + 2$

c) $y = \frac{1}{3}(4)^{x+1} + 1$

d) $n(s) = -\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{4}s} - 3$

Apply

7. Describe the transformations that must be applied to the graph of each exponential function $f(x)$ to obtain the transformed function. Write each transformed function in the form $y = a(c)^{b(x-h)} + k$.

a) $f(x) = \left(\frac{1}{2}\right)^x$, $y = f(x - 2) + 1$

b) $f(x) = 5^x$, $y = -0.5f(x - 3)$

c) $f(x) = \left(\frac{1}{4}\right)^x$, $y = -f(3x) + 1$

d) $f(x) = 4^x$, $y = 2f\left(-\frac{1}{3}(x - 1)\right) - 5$

8. For each pair of exponential functions in #7, sketch the original and transformed functions on the same set of coordinate axes. Explain your procedure.

9. The persistence of drugs in the human body can be modelled using an exponential function. Suppose a new drug follows the model $M(h) = M_0(0.79)^{\frac{h}{3}}$, where M is the mass, in milligrams, of drug remaining in the body; M_0 is the mass, in milligrams, of the dose taken; and h is the time, in hours, since the dose was taken.

a) Explain the roles of the numbers 0.79 and $\frac{1}{3}$.

b) A standard dose is 100 mg. Sketch the graph showing the mass of the drug remaining in the body for the first 48 h.

c) What does the M-intercept represent in this situation?

d) What are the domain and range of this function?

10. The rate at which liquids cool can be modelled by an approximation of Newton's law of cooling, $T(t) = (T_i - T_f)(0.9)^{\frac{t}{5}} + T_f$, where T_f represents the final temperature, in degrees Celsius; T_i represents the initial temperature, in degrees Celsius; and t represents the elapsed time, in minutes. Suppose a cup of coffee is at an initial temperature of 95 °C and cools to a temperature of 20 °C.

a) State the parameters a, b, h, and k for this situation. Describe the transformation that corresponds to each parameter.

b) Sketch a graph showing the temperature of the coffee over a period of 200 min.

c) What is the approximate temperature of the coffee after 100 min?

d) What does the horizontal asymptote of the graph represent?

11. A biologist places agar, a gel made from seaweed, in a Petri dish and infects it with bacteria. She uses the measurement of the growth ring to estimate the number of bacteria present. The biologist finds that the bacteria increase in population at an exponential rate of 20% every 2 days.

a) If the culture starts with a population of 5000 bacteria, what is the transformed exponential function in the form $P = a(c)^{bx}$ that represents the population, P, of the bacteria over time, x, in days?

b) Describe the parameters used to create the transformed exponential function.

c) Graph the transformed function and use it to predict the bacteria population after 9 days.

12. Living organisms contain carbon-12 (C-12), which does not decay, and carbon-14 (C-14), which does. When an organism dies, the amount of C-14 in its tissues decreases exponentially with a half-life of about 5730 years.

a) What is the transformed exponential function that represents the percent, P, of C-14 remaining after t years?

b) Graph the function and use it to determine the approximate age of a dead organism that has 20% of the original C-14 present in its tissues.

Did You Know?

Carbon dating can only be used to date organic material, or material from once-living things. It is only effective in dating organisms that lived up to about 60 000 years ago.

d) The values are undefined because they result in the square root of a negative number.

$f(x) = (-2)^x$ $\qquad$ $f(x) = (-2)^x$

$f\left(\frac{1}{2}\right) = (-2)^{\frac{1}{2}}$ $\qquad$ $f\left(\frac{5}{2}\right) = (-2)^{\frac{5}{2}}$

$f\left(\frac{1}{2}\right) = \sqrt{-2}$ $\qquad$ $f\left(\frac{5}{2}\right) = \sqrt{(-2)^5}$

e) Example: Exponential functions with positive bases result in smooth curves.

7.2 Transformations of Exponential Functions, pages 354 to 357

1. a) C **b)** D **c)** A **d)** B

2. a) D **b)** A **c)** B **d)** C

3. a) $a = 2$: vertical stretch by a factor of 2; $b = 1$: no horizontal stretch; $h = 0$: no horizontal translation; $k = -4$: vertical translation of 4 units down

b) $a = 1$: no vertical stretch; $b = 1$: no horizontal stretch; $h = 2$: horizontal translation of 2 units right; $k = 3$: vertical translation of 3 units up

c) $a = -4$: vertical stretch by a factor of 4 and a reflection in the x-axis; $b = 1$: no horizontal stretch; $h = -5$: horizontal translation of 5 units left; $k = 0$: no vertical translation

d) $a = 1$: no vertical stretch; $b = 3$: horizontal stretch by a factor of $\frac{1}{3}$; $h = 1$: horizontal translation of 1 unit right; $k = 0$: no vertical translation

e) $a = -\frac{1}{2}$: vertical stretch by a factor of $\frac{1}{2}$ and a reflection in the x-axis; $b = 2$: horizontal stretch by a factor of $\frac{1}{2}$; $h = 4$: horizontal translation of 4 units right; $k = 3$: vertical translation of 3 units up

f) $a = -1$: reflection in the x-axis; $b = 2$: horizontal stretch by a factor of $\frac{1}{2}$; $h = 1$: horizontal translation of 1 unit right; $k = 0$: no vertical translation

g) $a = 1.5$: vertical stretch by a factor of 1.5; $b = \frac{1}{2}$: horizontal stretch by a factor of 2; $h = 4$: horizontal translation of 4 units right; $k = -\frac{5}{2}$: vertical translation of $\frac{5}{2}$ units down

4. a) C: reflection in the x-axis, $a < 0$ and $0 < c < 1$, and vertical translation of 2 units up, $k = 2$

b) A: horizontal translation of 1 unit right, $h = 1$, and vertical translation of 2 units down, $k = -2$

c) D: reflection in the x-axis, $a < 0$ and $c > 1$, and vertical translation of 2 units up, $k = 2$

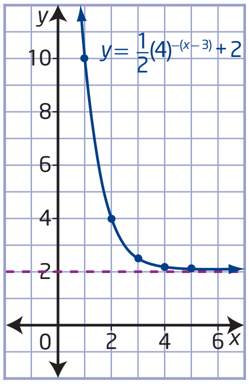
d) B: horizontal translation of 2 units right, $h = 2$, and vertical translation of 1 unit up, $k = 1$

5. a) $a = \frac{1}{2}$: vertical stretch by a factor of $\frac{1}{2}$; $b = -1$: reflection in the y-axis; $h = 3$: horizontal translation of 3 units right 3; $k = 2$: vertical translation of 2 units up

b)

$y = 4^x$	$y = 4^{-x}$	$y = \frac{1}{2}(4)^{-x}$	$y = \frac{1}{2}(4)^{-(x-3)} + 2$
$\left(-2, \frac{1}{16}\right)$	$\left(2, \frac{1}{16}\right)$	$\left(2, \frac{1}{32}\right)$	$\left(5, \frac{65}{32}\right)$
$\left(-1, \frac{1}{4}\right)$	$\left(1, \frac{1}{4}\right)$	$\left(1, \frac{1}{8}\right)$	$\left(4, \frac{17}{8}\right)$
(0, 1)	(0, 1)	$\left(0, \frac{1}{2}\right)$	$\left(3, \frac{5}{2}\right)$
(1, 4)	(−1, 4)	(−1,2)	(2, 4)
(2, 16)	(−2, 16)	(−2, 8)	(1, 10)

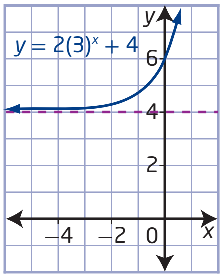
c)



d) domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y > 2, y \in \mathrm{R}\}$, horizontal asymptote $y = 2$, y-intercept 34

6. a) i), ii) $a = 2$: vertical stretch by a factor of 2; $b = 1$: no horizontal stretch; $h = 0$: no horizontal translation; $k = 4$: vertical translation of 4 units up

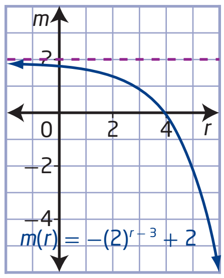
iii)



iv) domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y > 4, y \in \mathrm{R}\}$, horizontal asymptote $y = 4$, y-intercept 6

b) i), ii) $a = -1$: reflection in the x-axis; $b = 1$: no horizontal stretch; $h = 3$: horizontal translation of 3 units right; $k = 2$: vertical translation of 2 units up

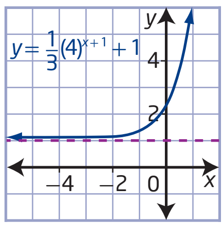
iii)



iv) domain $\{r \mid r \in \mathrm{R}\}$, range $\{m \mid m < 2, m \in \mathrm{R}\}$, horizontal asymptote $m = 2$, m-intercept $\frac{15}{8}$, r-intercept 4

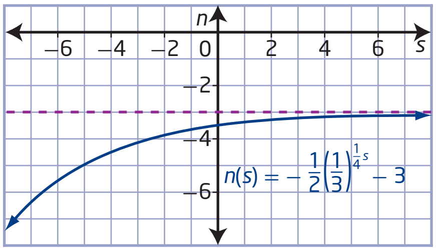
c) i), ii) $a = \frac{1}{3}$: vertical stretch by a factor of $\frac{1}{3}$; $b = 1$: no horizontal stretch; $h = -1$: horizontal translation of 1 unit left; $k = 1$: vertical translation of 1 unit up

iii)



iv) domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y > 1, y \in \mathrm{R}\}$, horizontal asymptote $y = 1$, y-intercept $\frac{7}{3}$

d) i), ii) $a = -\frac{1}{2}$: vertical stretch by a factor of $\frac{1}{2}$ and a reflection in the x-axis; $b = \frac{1}{4}$: horizontal stretch by a factor of 4; $h = 0$: no horizontal translation; $k = -3$: vertical translation of 3 units down

iii) 

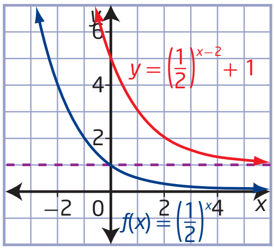
iv) domain $\{s \mid s \in \mathrm{R}\}$, range $\{n \mid n < -3, n \in \mathrm{R}\}$, horizontal asymptote $n = -3$, n-intercept $-\frac{7}{2}$

7. a) horizontal translation of 2 units right and vertical translation of 1 unit up; $y = \left(\frac{1}{2}\right)^{x-2} + 1$

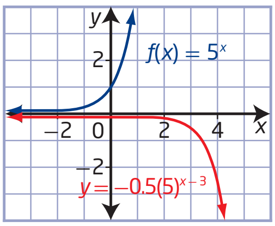
b) reflection in the x-axis, vertical stretch by a factor of 0.5, and horizontal translation of 3 units right; $y = -0.5(5)^{x-3}$

c) reflection in the x-axis, horizontal stretch by a factor of $\frac{1}{3}$, and vertical translation of 1 unit up; $y = -\left(\frac{1}{4}\right)^{3x} + 1$

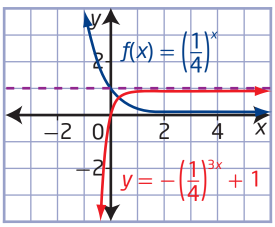
d) vertical stretch by a factor of 2, reflection in the y-axis, horizontal stretch by a factor of 3, horizontal translation of 1 unit right, and vertical translation of 5 units down; $y = 2(4)^{-\frac{1}{3}(x-1)} - 5$

8. a) 

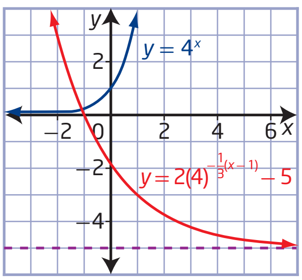
Map all points (x, y) on the graph of $f(x)$ to $(x + 2, y + 1)$.

b) 

Map all points (x, y) on the graph of $f(x)$ to $(x + 3, -0.5y)$.

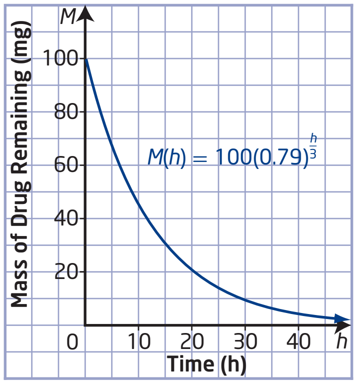
c) 

Map all points (x, y) on the graph of $f(x)$ to $\left(\frac{1}{3}x, -y + 1\right)$.

d) 

Map all points (x, y) on the graph of $f(x)$ to $(-3x + 1, 2y - 5)$.

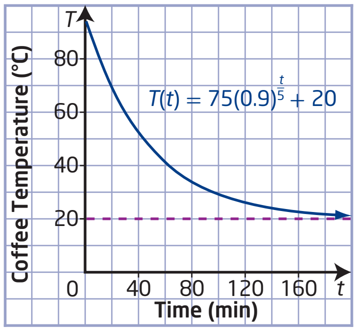
9. a) 0.79 represents the 79% of the drug remaining in exponential decay after $\frac{1}{3}$ h.

b) 

c) The M-intercept represents the drug dose taken.

d) domain $\{h \mid h \geq 0, h \in \mathrm{R}\}$, range $\{M \mid 0 < M \leq 100, M \in \mathrm{R}\}$

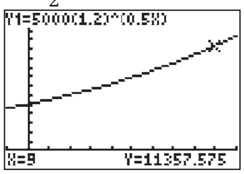
10. a) $a = 75$: vertical stretch by a factor of 75;
$b = \frac{1}{5}$: horizontal stretch by a factor of 5;
$h = 0$: no horizontal translation;
$k = 20$: vertical translation of 20 units up

b) 

c) 29.1 °C **d)** final temperature of the coffee

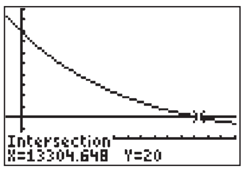
11. a) $P = 5000(1.2)^{\frac{1}{2}x}$

b) $a = 5000$: vertical stretch by a factor of 5000;
$b = \frac{1}{2}$: horizontal stretch by a factor of 2

c) 

approximately 11 357 bacteria

12. a) $P = 100\left(\frac{1}{2}\right)^{\frac{t}{5730}}$

b) 

approximately 13 305 years old

13. a) 527.8 cm² **b)** 555 h

14. a) 1637 foxes

b) Example: Disease or lack of food can change the rate of growth of the foxes. Exponential growth suggests that the population will grow without bound, and therefore the fox population will grow beyond the possible food sources, which is not good if not controlled.