Key Ideas

- When solving a logarithmic equation algebraically, start by applying the laws of logarithms to express one side or both sides of the equation as a single logarithm.
- Some useful properties are listed below, where c, L, $R > 0$ and $c \neq 1$.
  - If $\log_c L = \log_c R$, then $L = R$.
  - The equation $\log_c L = R$ can be written with logarithms on both sides of the equation as $\log_c L = \log_c c^R$.
  - The equation $\log_c L = R$ can be written in exponential form as $L = c^R$.
  - The logarithm of zero or a negative number is undefined. To identify whether a root is extraneous, substitute the root into the original equation and check whether all of the logarithms are defined.
- You can solve an exponential equation algebraically by taking logarithms of both sides of the equation. If $L = R$, then $\log_c L = \log_c R$, where c, L, $R > 0$ and $c \neq 1$. Then, apply the power law for logarithms to solve for an unknown.
- You can solve an exponential equation or a logarithmic equation using graphical methods.
- Many real-world situations can be modelled with an exponential or a logarithmic equation. A general model for many problems involving exponential growth or decay is

  $$\text{final quantity} = \text{initial quantity} \times (\text{change factor})^{\text{number of changes}}$$

Check Your Understanding

Practise

1. Solve. Give exact answers.

a) $15 = 12 + \log x$

b) $\log_5 (2x - 3) = 2$

c) $4 \log_3 x = \log_3 81$

d) $2 = \log (x - 8)$

2. Solve for x. Give your answers to two decimal places.

a) $4(7^x) = 92$

b) $2^{\frac{x}{3}} = 11$

c) $6^{x-1} = 271$

d) $4^{2x+1} = 54$

3. Hamdi algebraically solved the equation $\log_3 (x - 8) - \log_3 (x - 6) = 1$ and found $x = 5$ as a possible solution. The following shows Hamdi's check for $x = 5$.

Left Side

$\log_3 \frac{x - 8}{x - 6}$

$= \log_3 \frac{5 - 8}{5 - 6}$

$= \log_3 3$

$= 1$

Right Side

1

Left Side = Right Side

Do you agree with Hamdi's check? Explain why or why not.

4. Determine whether the possible roots listed are extraneous to the logarithmic equation given.

a) $\log_7 x + \log_7 (x - 1) = \log_7 4x$
possible roots: $x = 0$, $x = 5$

b) $\log_6 (x^2 - 24) - \log_6 x = \log_6 5$
possible roots: $x = 3$, $x = -8$

c) $\log_3 (x + 3) + \log_3 (x + 5) = 1$
possible roots: $x = -2$, $x = -6$

d) $\log_2 (x - 2) = 2 - \log_2 (x - 5)$
possible roots: $x = 1$, $x = 6$

5. Solve for x.

a) $2 \log_3 x = \log_3 32 + \log_3 2$

b) $\frac{3}{2} \log_7 x = \log_7 125$

c) $\log_2 x - \log_2 3 = 5$

d) $\log_6 x = 2 - \log_6 4$

Apply

6. Three students each attempted to solve a different logarithmic equation. Identify and describe any error in each person's work, and then correctly solve the equation.

a) *Rubina's work:*

$\log_6 (2x + 1) - \log_6 (x - 1) = \log_6 5$
$\log_6 (x + 2) = \log_6 5$
$x + 2 = 5$
$x = 3$

The solution is $x = 3$.

b) *Ahmed's work:*

$2 \log_5 (x + 3) = \log_5 9$
$\log_5 (x + 3)^2 = \log_5 9$
$(x + 3)^2 = 9$
$x^2 + 6x + 9 = 9$
$x(x + 6) = 0$
$x = 0$ or $x = -6$

There is no solution.

c) *Jennifer's work:*

$\log_2 x + \log_2 (x + 2) = 3$
$\log_2 (x(x + 2)) = 3$
$x(x + 2) = 3$
$x^2 + 2x - 3 = 0$
$(x + 3)(x - 1) = 0$
$x = -3$ or $x = 1$

The solution is $x = 1$.

7. Determine the value of x. Round your answers to two decimal places.

a) $7^{2x} = 2^{x+3}$

b) $1.6^{x-4} = 5^{3x}$

c) $9^{2x-1} = 71^{x+2}$

d) $4(7^{x+2}) = 9^{2x-3}$

8. Solve for x.

a) $\log_5 (x - 18) - \log_5 x = \log_5 7$

b) $\log_2 (x - 6) + \log_2 (x - 8) = 3$

c) $2 \log_4 (x + 4) - \log_4 (x + 12) = 1$

d) $\log_3 (2x - 1) = 2 - \log_3 (x + 1)$

e) $\log_2 \sqrt{x^2 + 4x} = \frac{5}{2}$

9. The apparent magnitude of a celestial object is how bright it appears from Earth. The absolute magnitude is its brightness as it would seem from a reference distance of 10 parsecs (pc). The difference between the apparent magnitude, m, and the absolute magnitude, M, of a celestial object can be found using the equation $m - M = 5 \log d - 5$, where d is the distance to the celestial object, in parsecs. Sirius, the brightest star visible at night, has an apparent magnitude of -1.44 and an absolute magnitude of 1.45.

a) How far is Sirius from Earth in parsecs?

b) Given that 1 pc is approximately 3.26 light years, what is the distance in part a) in light years?

10. Small animal characters in animated features are often portrayed with big endearing eyes. In reality, the eye size of many vertebrates is related to body mass by the logarithmic equation $\log E = \log 10.61 + 0.1964 \log m$, where E is the eye axial length, in millimetres, and m is the body mass, in kilograms. To the nearest kilogram, predict the mass of a mountain goat with an eye axial length of 24 mm.

11. A remote lake that previously contained no northern pike is stocked with these fish. The population, P, of northern pike after t years can be determined by the equation $P = 10\ 000(1.035)^t$.

a) How many northern pike were put into the lake when it was stocked?

b) What is the annual growth rate, as a percent?

c) How long will it take for the number of northern pike in the lake to double?

12. The German astronomer Johannes Kepler developed three major laws of planetary motion. His third law can be expressed by the equation $\log T = \frac{3}{2} \log d - 3.263$, where T is the time, in Earth years, for the planet to revolve around the sun and d is the average distance, in millions of kilometres, from the sun.

a) Pluto is on average 5906 million kilometres from the sun. To the nearest Earth year, how long does it take Pluto to revolve around the sun?

b) Mars revolves around the sun in 1.88 Earth years. How far is Mars from the sun, to the nearest million kilometres?

13. The compound interest formula is $A = P(1 + i)^n$, where A is the future amount, P is the present amount or principal, i is the interest rate per compounding period expressed as a decimal, and n is the number of compounding periods. All interest rates are annual percentage rates (APR).

a) David inherits \$10 000 and invests in a guaranteed investment certificate (GIC) that earns 6%, compounded semi-annually. How long will it take for the GIC to be worth \$11 000?

b) Linda used a credit card to purchase a \$1200 laptop computer. The rate of interest charged on the overdue balance is 28% per year, compounded daily. How many days is Linda's payment overdue if the amount shown on her credit card statement is \$1241.18?

c) How long will it take for money invested at 5.5%, compounded semi-annually, to triple in value?

14. A mortgage is a long-term loan secured by property. A mortgage with a present value of \$250 000 at a 7.4% annual percentage rate requires semi-annual payments of \$10 429.01 at the end of every 6 months. The formula for the present value, PV, of the mortgage is $PV = \frac{R[1 - (1 + i)^{-n}]}{i}$, where n is the number of equal periodic payments of R dollars and i is the interest rate per compounding period, as a decimal. After how many years will the mortgage be completely paid off?

15. Swedish researchers report that they have discovered the world's oldest living tree. The spruce tree's roots were radiocarbon dated and found to have 31.5% of their carbon-14 (C-14) left. The half-life of C-14 is 5730 years. How old was the tree when it was discovered?

Norway spruce, Dalarna, Sweden

19. a) $y = \log_c x$
$c^y = x$
$\log_d c^y = \log_d x$
$y \log_d c = \log_d x$
$y = \dfrac{\log_d x}{\log_d c}$

b) 3.2479

c) $\varphi = -\dfrac{\log D}{\log 2}$

d) 207.9 times larger

20. a) Left Side
$= \log_{q^3} p^3$
$= \dfrac{\log_q p^3}{\log_q q^3}$
$= \dfrac{3 \log_q p}{3 \log_q q}$
$= \dfrac{\log_q p}{1}$
$=$ Right Side

b) Left Side
$= \dfrac{1}{\log_p 2} - \dfrac{1}{\log_q 2}$
$= \dfrac{1}{\frac{\log_2 2}{\log_2 p}} - \dfrac{1}{\frac{\log_2 2}{\log_2 q}}$
$= \dfrac{\log_2 p}{\log_2 2} - \dfrac{\log_2 q}{\log_2 2}$
$= \dfrac{\log_2 p - \log_2 q}{\log_2 2}$
$= \log_2 \dfrac{p}{q}$
$=$ Right Side

c)

Left Side	Right Side
$= \dfrac{1}{\log_q p} + \dfrac{1}{\log_q p}$	$= \dfrac{1}{\log_{q^2} p}$
$= \dfrac{1}{\frac{\log p}{\log q}} + \dfrac{1}{\frac{\log q}{\log p}}$	$= \dfrac{1}{\log p / \log q^2}$
$= \dfrac{\log q}{\log p} + \dfrac{\log q}{\log p}$	$= \dfrac{\log q^2}{\log p}$
$= \dfrac{2 \log q}{\log p}$	$= \dfrac{2 \log q}{\log p}$

Left Side = Right Side

d) Left Side $= \log_{\frac{1}{q}} p$
$= \dfrac{\log_q p}{\log_q q^{-1}}$
$= -\log_q p$
$= \log_q \dfrac{1}{p}$
$=$ Right Side

C1 a) Stretch vertically about the x-axis by a factor of 3.

b) Stretch vertically about the x-axis by a factor of 5 and translate 2 units left.

c) Reflect in the x-axis.

d) Reflect in the x-axis, stretch vertically about the x-axis by a factor of $\frac{1}{2}$, and translate 6 units right.

C2 -1

C3 a) log 2 **b)** 15 log 2

C4 Answers will vary.

8.4 Logarithmic and Exponential Equations, pages 412 to 415

1. a) 1000 **b)** 14 **c)** 3 **d)** 108

2. a) 1.61 **b)** 10.38 **c)** 4.13 **d)** 0.94

3. No, since $\log_3 (x - 8)$ and $\log_3 (x - 6)$ are not defined when $x = 5$.

4. a) $x = 0$ is extraneous.

b) Both roots are extraneous.

c) $x = -6$ is extraneous.

d) $x = 1$ is extraneous.

5. a) $x = 8$ **b)** $x = 25$ **c)** $x = 96$ **d)** $x = 9$

6. a) Rubina subtracted the contents of the log when she should have divided them. The solution should be
$\log_6 \left(\dfrac{2x + 1}{x - 1}\right) = \log_6 5$
$2x + 1 = 5(x - 1)$
$1 + 5 = 5x - 2x$
$6 = 3x$
$x = 2$

b) Ahmed incorrectly concluded that there was no solution. The solution is $x = 0$.

c) Jennifer incorrectly eliminated the log in the third line. The solution, from the third line on, should be
$x(x + 2) = 2^3$
$x^2 + 2x - 8 = 0$
$(x - 2)(x + 4) = 0$
So, $x = 2$ or $x = -4$.
Since $x > 0$, the solution is $x = 2$.

7. a) 0.65 **b)** -0.43 **c)** 81.37 **d)** 4.85

8. a) no solution ($x = -3$ not possible)

b) $x = 10$ **c)** $x = 4$ **d)** $x = 2$ **e)** $x = -8, 4$

9. a) about 2.64 pc **b)** about 8.61 light years

10. 64 kg

11. a) 10 000 **b)** 3.5%

c) approximately 20.1 years

12. a) 248 Earth years **b)** 228 million kilometres

13. a) 2 years **b)** 44 days **c)** 20.5 years

14. 30 years

15. approximately 9550 years

16. 8 days

17. 34.0 m

18. $x = 4.5$, $y = 0.5$

19. a) The first line is not true.

b) To go from line 4 to line 5, you are dividing by a negative quantity, so the inequality sign must change direction.

20. a) $x = 100$ **b)** $x = \dfrac{1}{100}, 100$ **c)** $x = 1, 100$

21. a) $x = 16$ **b)** $x = 9$

22. $x = -5, 2, 4$

C1 a) $\log 8 + \log 2^x = \log 512$
$x \log 2 = \log 512 - \log 8$
$x \log 2 = \log 64$
$x = 6$

b) She could have divided by 8 as the first step.

c) Answers will vary.

C2 12

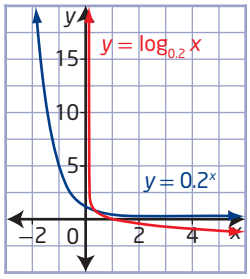
C3 14

C4 a) $x = \dfrac{\pi}{4}, \dfrac{7\pi}{4}$ **b)** $x = \dfrac{\pi}{2}$

C5 Answers will vary.

Chapter 8 Review, pages 416 to 418

1. a)



b) i) domain $\{x \mid x > 0, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$

ii) x-intercept 1

iii) no y-intercept

iv) vertical asymptote $x = 0$

c) $y = \log_{0.2} x$

2. $c = 4$

3. $2^4 = 16$ and $2^5 = 32$, so the answer must be between 4 and 5.