Check Your Understanding

Practise

1. Write each expression as a single trigonometric function.

a) $\cos 43° \cos 27° - \sin 43° \sin 27°$

b) $\sin 15° \cos 20° + \cos 15° \sin 20°$

c) $\cos^2 19° - \sin^2 19°$

d) $\sin \frac{3\pi}{2} \cos \frac{5\pi}{4} - \cos \frac{3\pi}{2} \sin \frac{5\pi}{4}$

e) $8 \sin \frac{\pi}{3} \cos \frac{\pi}{3}$

2. Simplify and then give an exact value for each expression.

a) $\cos 40° \cos 20° - \sin 40° \sin 20°$

b) $\sin 20° \cos 25° + \cos 20° \sin 25°$

c) $\cos^2 \frac{\pi}{6} - \sin^2 \frac{\pi}{6}$

d) $\cos \frac{\pi}{2} \cos \frac{\pi}{3} - \sin \frac{\pi}{2} \sin \frac{\pi}{3}$

3. Using only one substitution, which form of the double-angle identity for cosine will simplify the expression $1 - \cos 2x$ to one term? Show how this happens.

4. Write each expression as a single trigonometric function.

a) $2 \sin \frac{\pi}{4} \cos \frac{\pi}{4}$

b) $(6 \cos^2 24° - 6 \sin^2 24°) \tan 48°$

c) $\frac{2 \tan 76°}{1 - \tan^2 76°}$

d) $2 \cos^2 \frac{\pi}{6} - 1$

e) $1 - 2 \cos^2 \frac{\pi}{12}$

5. Simplify each expression to a single primary trigonometric function.

a) $\frac{\sin 2\theta}{2 \cos \theta}$

b) $\cos 2x \cos x + \sin 2x \sin x$

c) $\frac{\cos 2\theta + 1}{2 \cos \theta}$

d) $\frac{\cos^3 x}{\cos 2x + \sin^2 x}$

6. Show using a counterexample that the following is not an identity: $\sin (x - y) = \sin x - \sin y$.

7. Simplify $\cos (90° - x)$ using a difference identity.

8. Determine the exact value of each trigonometric expression.

a) $\cos 75°$

b) $\tan 165°$

c) $\sin \frac{7\pi}{12}$

d) $\cos 195°$

e) $\csc \frac{\pi}{12}$

f) $\sin \left(-\frac{\pi}{12}\right)$

Apply

Yukon River at Whitehorse

9. On the winter solstice, December 21 or 22, the power, P, in watts, received from the sun on each square metre of Earth can be determined using the equation
$P = 1000 (\sin x \cos 113.5° + \cos x \sin 113.5°)$, where x is the latitude of the location in the northern hemisphere.

a) Use an identity to write the equation in a more useful form.

b) Determine the amount of power received at each location.

i) Whitehorse, Yukon, at 60.7° N

ii) Victoria, British Columbia, at 48.4° N

iii) Igloolik, Nunavut, at 69.4° N

c) Explain the answer for part iii) above. At what latitude is the power received from the sun zero?

10. Simplify $\cos(\pi + x) + \cos(\pi - x)$.

11. Angle θ is in quadrant II and $\sin\theta = \frac{5}{13}$. Determine an exact value for each of the following.

a) $\cos 2\theta$

b) $\sin 2\theta$

c) $\sin\left(\theta + \frac{\pi}{2}\right)$

12. The double-angle identity for tangent in terms of the tangent function is $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$.

a) Verify numerically that this equation is true for $x = \frac{\pi}{6}$.

b) The expression $\tan 2x$ can also be written using the quotient identity for tangent: $\tan 2x = \frac{\sin 2x}{\cos 2x}$. Verify this equation numerically when $x = \frac{\pi}{6}$.

c) The expression $\frac{\sin 2x}{\cos 2x}$ from part b) can be expressed as $\frac{2\sin x\cos x}{\cos^2 x - \sin^2 x}$ using double-angle identities. Show how the expression for $\tan 2x$ used in part a) can also be rewritten in the form $\frac{2\sin x\cos x}{\cos^2 x - \sin^2 x}$.

13. The horizontal distance, d, in metres, travelled by a ball that is kicked at an angle, θ, with the ground is modelled by the formula $d = \frac{2(v_0)^2\sin\theta\cos\theta}{g}$, where v_0 is the initial velocity of the ball, in metres per second, and g is the force of gravity ($9.8\ m/s^2$).

a) Rewrite the formula using a double-angle identity.

b) Determine the angle $\theta \in (0°, 90°)$ that would result in a maximum distance for an initial velocity v_0.

c) Explain why it might be easier to answer part b) with the double-angle version of the formula that you determined in part a).

14. If $(\sin x + \cos x)^2 = k$, then what is the value of $\sin 2x$ in terms of k?

15. Show that each expression can be simplified to $\cos 2x$.

a) $\cos^4 x - \sin^4 x$

b) $\frac{\csc^2 x - 2}{\csc^2 x}$

16. Simplify each expression to the equivalent expression shown.

a) $\frac{1 - \cos 2x}{2}$ $\quad \sin^2 x$

b) $\frac{4 - 8\sin^2 x}{2\sin x\cos x}$ $\quad \frac{4}{\tan 2x}$

17. If the point (2, 5) lies on the terminal arm of angle x in standard position, what is the value of $\cos(\pi + x)$?

18. What value of k makes the equation $\sin 5x\cos x + \cos 5x\sin x = 2\sin kx\cos kx$ true?

19. a) If $\cos\theta = \frac{3}{5}$ and $0 < \theta < 2\pi$, determine the value(s) of $\sin\left(\theta + \frac{\pi}{6}\right)$.

b) If $\sin\theta = -\frac{2}{3}$ and $\frac{3\pi}{2} < \theta < 2\pi$, determine the value(s) of $\cos\left(\theta + \frac{\pi}{3}\right)$.

20. If ∠A and ∠B are both in quadrant I, and $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, evaluate each of the following,

a) $\cos(A - B)$

b) $\sin(A + B)$

c) $\cos 2A$

d) $\sin 2A$

Extend

21. Determine the missing primary trigonometric ratio that is required for the expression $\frac{\blacksquare\sin 2x}{2 - 2\cos^2 x}$ to simplify to

a) $\cos x$

b) 1

22. Use a double-angle identity for cosine to determine the half-angle formula for cosine, $\cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$.

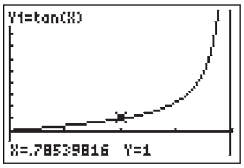
16. $\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} = \frac{1-\sin\theta+1+\sin\theta}{(1-\sin\theta)(1+\sin\theta)}$
$= \frac{2}{(1-\sin^2\theta)}$
$= 2\sec^2\theta$

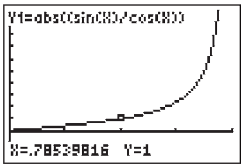
17. $m = \csc x$

C1 $\cot^2 x + 1$
$= \frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x}$
$= \frac{\cos^2 x + \sin^2 x}{\sin^2 x}$
$= \frac{1}{\sin^2 x}$
$= \csc^2 x$

C2 $\left(\frac{\sin\theta}{1+\cos\theta}\right)\left(\frac{1-\cos\theta}{1-\cos\theta}\right)$
$= \frac{\sin\theta - \sin\theta\cos\theta}{1-\cos^2\theta}$
$= \frac{\sin\theta - \sin\theta\cos\theta}{\sin^2\theta}$
$= \frac{1-\cos\theta}{\sin\theta}$
It helps to simplify by creating an opportunity to use the Pythagorean identity.

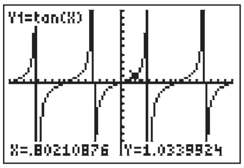
C3 **Step 1**

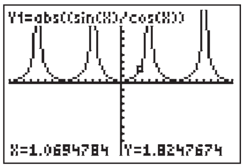




Yes, over this domain it is an identity.

Step 2





The equation is not an identity since the graphs of the two sides are not the same.

Step 3 Example: $y = \cot\theta$ and $y = \left|\frac{\cos\theta}{\sin\theta}\right|$ are identities over the domain $0 \le \theta \le \frac{\pi}{2}$ but not over the domain $-2\pi \le \theta \le 2\pi$

Step 4 The weakness with this approach is that for some more complicated identities you may think it is an identity when really it is only an identity over that domain.

6.2 Sum, Difference, and Double-angle Identities, pages 306 to 308

1. a) $\cos 70°$ b) $\sin 35°$ c) $\cos 38°$
d) $\sin\frac{\pi}{4}$ e) $4\sin\frac{2\pi}{3}$

2. a) $\cos 60° = 0.5$ b) $\sin 45° = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$
c) $\cos\frac{\pi}{3} = 0.5$ d) $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

3. $\cos 2x = 1 - 2\sin^2 x$;
$1 - \cos 2x = 1 - 1 + 2\sin^2 x = 2\sin^2 x$

4. a) $\sin\frac{\pi}{2}$ b) $6\sin 48°$ c) $\tan 152°$ d) $\cos\frac{\pi}{3}$
e) $-\cos\frac{\pi}{6}$

5. a) $\sin\theta$ b) $\cos x$ c) $\cos\theta$ d) $\cos x$

6. Example: When $x = 60°$ and $y = 30°$, then left side $= 0.5$, but right side ≈ 0.366.

7. $\cos(90° - x) = \cos 90°\cos x + \sin 90°\sin x$
$= \sin x$

8. a) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ or $\frac{\sqrt{6}-\sqrt{2}}{4}$ b) $\frac{-\sqrt{3}+1}{\sqrt{3}+1}$ or $\sqrt{3}-2$
c) $\frac{1+\sqrt{3}}{2\sqrt{2}}$ or $\frac{\sqrt{2}+\sqrt{6}}{4}$ d) $\frac{-\sqrt{3}-1}{2\sqrt{2}}$ or $\frac{-\sqrt{6}-\sqrt{2}}{4}$
e) $\sqrt{2}(1+\sqrt{3})$ f) $\frac{1-\sqrt{3}}{2\sqrt{2}}$ or $\frac{\sqrt{2}-\sqrt{6}}{4}$

9. a) $P = 1000\sin(x + 113.5°)$
b) i) 101.056 W/m² ii) 310.676 W/m²
iii) −50.593 W/m²
c) The answer in part iii) is negative which means that there is no sunlight reaching Igloolik. At latitude 66.5°, the power received is 0 W/m².

10. $-2\cos x$

11. a) $\frac{119}{169}$ b) $-\frac{120}{169}$ c) $-\frac{12}{13}$

12. a) Both sides are equal for this value.
b) Both sides are equal for this value.
c) $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$
$= \frac{2\tan x}{1-\tan^2 x}\left(\frac{\cos^2 x}{\cos^2 x}\right)$
$= \frac{2\left(\frac{\sin x}{\cos x}\right)(\cos^2 x)}{\left(1-\frac{\sin^2 x}{\cos^2 x}\right)\cos^2 x}$
$= \frac{2\sin x\cos x}{\cos^2 x - \sin^2 x}$

13. a) $d = \frac{v_o^2\sin 2\theta}{g}$ b) 45°
c) It is easier after applying the double-angle identity since there is only one trigonometric function whose value has to be found.

14. $k - 1$

15. a) $\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$
$= \cos^2 x - \sin^2 x$
$= \cos 2x$
b) $\frac{\csc^2 x - 2}{\csc^2 x} = 1 - \frac{2}{\csc^2 x}$
$= 1 - 2\sin^2 x$
$= \cos 2x$

16. a) $\frac{1-\cos 2x}{2} = \frac{1-1+2\sin^2 x}{2} = \sin^2 x$
b) $\frac{4-8\sin^2 x}{2\sin x\cos x} = \frac{4\cos 2x}{\sin 2x} = \frac{4}{\tan 2x}$

17. $-\frac{2}{\sqrt{29}}$

18. $k = 3$

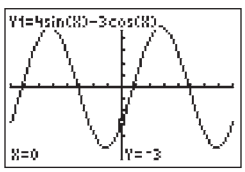
19. a) $0.9928, -0.392\,82$ or $\frac{\pm 4\sqrt{3}+3}{10}$
b) 0.9500 or $\frac{\sqrt{5}+2\sqrt{3}}{6}$

20. a) $\frac{56}{65}$ b) $\frac{63}{65}$ c) $\frac{-7}{25}$ d) $\frac{24}{25}$

21. a) $\sin x$ b) $\tan x$

22. $\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1$
$\frac{\cos x + 1}{2} = \cos^2\left(\frac{x}{2}\right)$
$\pm\sqrt{\frac{\cos x + 1}{2}} = \cos\frac{x}{2}$

23. a)



b) $a = 5$, $c = 37°$
c) $y = 5\sin(x - 36.87°)$