Key Ideas

- The fundamental counting principle can be used to determine the number of different arrangements. If one task can be performed in a ways, a second task in b ways, and a third task in c ways, then all three tasks can be arranged in $a \times b \times c$ ways.
- Factorial notation is an abbreviation for products of successive positive integers.

  $5! = (5)(4)(3)(2)(1)$
  $(n + 1)! = (n + 1)(n)(n - 1)(n - 2)\cdots(3)(2)(1)$
- A permutation is an arrangement of objects in a definite order. The number of permutations of n different objects taken r at a time is given by ${}_nP_r = \frac{n!}{(n - r)!}$.
- A set of n objects containing a identical objects of one kind, b identical objects of another kind, and so on, can be arranged in $\frac{n!}{a!b!\ldots}$ ways.
- Some problems have more than one case. One way to solve such problems is to establish cases that together cover all of the possibilities. Calculate the number of arrangements for each case and then add the values for all cases to obtain the total number of arrangements.

Check Your Understanding

Practise

1. Use an organized list or a tree diagram to identify the possible arrangements for

a) the ways that three friends, Jo, Amy, and Mike, can arrange themselves in a row.

b) the ways that you can arrange the digits 2, 5, 8, and 9 to form two-digit numbers.

c) the ways that a customer can choose a starter, a main course, and a dessert from the following menu.

LUNCH SPECIAL MENU
Starter: *soup or salad*
Main: *chili or hamburger or chicken or fish*
Dessert: *ice cream or fruit salad*

2. Evaluate each expression.

a) ${}_8P_2$ **b)** ${}_7P_5$

c) ${}_6P_6$ **d)** ${}_4P_1$

3. Show that $4! + 3! \neq (4 + 3)!$.

4. What is the value of each expression?

a) $9!$ **b)** $\frac{9!}{5!4!}$

c) $(5!)(3!)$ **d)** $6(4!)$

e) $\frac{102!}{100!2!}$ **f)** $7! - 5!$

5. In how many different ways can you arrange all of the letters of each word?

a) hoodie **b)** decided

c) aqilluqqaaq **d)** deeded

e) puppy **f)** baguette

Did You Know?

The Inuit have many words to describe snow. The word *aqilluqqaaq* means fresh and soggy snow in one dialect of Inuktitut.

6. Four students are running in an election for class representative on the student council. In how many different ways can the four names be listed on the ballot?

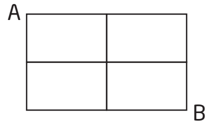
7. Solve for the variable.

a) ${}_{n}P_{2} = 30$ **b)** ${}_{n}P_{3} = 990$

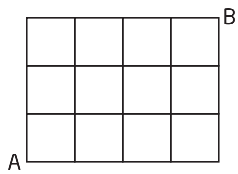
c) ${}_{6}P_{r} = 30$ **d)** $2({}_{n}P_{2}) = 60$

8. Determine the number of pathways from A to B.

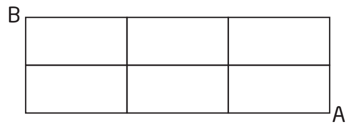
a) Move only down or to the right.



b) Move only up or to the right.



c) Move only up or to the left.



9. Describe the cases you could use to solve each problem. Do not solve.

a) How many 3-digit even numbers greater than 200 can you make using the digits 1, 2, 3, 4, and 5?

b) How many four-letter arrangements beginning with either B or E and ending with a vowel can you make using the letters A, B, C, E, U, and G?

10. In how many ways can four girls and two boys be arranged in a row if

a) the boys are on each end of the row?

b) the boys must be together?

c) the boys must be together in the middle of the row?

11. In how many ways can seven books be arranged on a shelf if

a) the books are all different?

b) two of the books are identical?

c) the books are different and the mathematics book must be on an end?

d) the books are different and four particular books must be together?

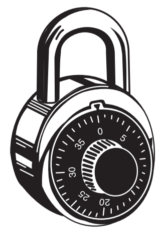
Apply

12. How many six-letter arrangements can you make using all of the letters A, B, C, D, E, and F, without repetition? Of these, how many begin and end with a consonant?

13. A national organization plans to issue its members a 4-character ID code. The first character can be any letter other than O. The last 3 characters are to be 3 different digits. If the organization has 25 300 members, will they be able to assign each member a different ID code? Explain.

14. Iblauk lives in Baker Lake, Nunavut. She makes oven mitts to sell. She has wool duffel in red, dark blue, green, light blue, and yellow for the body of each mitt. She has material for the wrist edge in dark green, pink, royal blue, and red. How many different colour combinations of mitts can Iblauk make?

15. You have forgotten the number sequence to your lock. You know that the correct code is made up of three numbers (right-left-right). The numbers can be from 0 to 39 and repetitions are allowed. If you can test one number sequence every 15 s, how long will it take to test all possible number sequences? Express your answer in hours.



16. Jodi is parking seven different types of vehicles side by side facing the display window at the dealership where she works.

a) In how many ways can she park the vehicles?

b) In how many ways can she park them so that the pickup truck is next to the hybrid car?

c) In how many ways can she park them so that the convertible is not next to the subcompact?

17. a) How many arrangements using all of the letters of the word *parallel* are possible?

b) How many of these arrangements have all of the *l*'s together?

18. The number of different permutations using all of the letters in a particular set is given by $\frac{5!}{2!2!}$.

a) Create a set of letters for which this is true.

b) What English word could have this number of arrangements of its letters?

19. How many integers from 3000 to 8999, inclusive, contain no 7s?

20. Postal codes in Canada consist of three letters and three digits. Letters and digits alternate, as in the code R7B 5K1.

a) How many different postal codes are possible with this format?

b) Do you think Canada will run out of postal codes? Why or why not?

Did You Know?

The Canadian postal code system was established in 1971. The first letters of the codes are assigned to provinces and territories from east to west:

A = Newfoundland and Labrador

...

Y = Yukon Territory

Some provinces have more than one letter, such as H and J for Québec. Some letters, such as I, are not currently used.

21. *Cent mille milliards de poèmes* (*One Hundred Million Million Poems*) was written in 1961 by Raymond Queneau, a French poet, novelist, and publisher. The book is 10 pages long, with 1 sonnet per page. A sonnet is a poem with 14 lines. Each line of every sonnet can be replaced by a line at the same position on a different page. Regardless of which lines are used, the poem makes sense.

a) How many arrangements of the lines are possible for one sonnet?

b) Is the title of the book of poems reasonable? Explain.

22. Use your understanding of factorial notation and the symbol ${}_nP_r$ to solve each equation.

a) ${}_3P_r = 3!$

b) ${}_7P_r = 7!$

c) ${}_nP_3 = 4({}_{n-1}P_2)$

d) $n({}_5P_3) = {}_7P_5$

23. Use ${}_nP_n$ to show that $0! = 1$.

24. Explain why ${}_3P_5$ gives an error message when evaluated on a calculator.

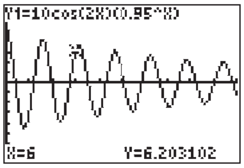
25. How many odd numbers of at most three digits can be formed using the digits 0, 1, 2, 3, 4, and 5 without repetitions?

26. How many even numbers of at least four digits can be formed using the digits 0, 1, 2, 3, and 5 without repetitions?

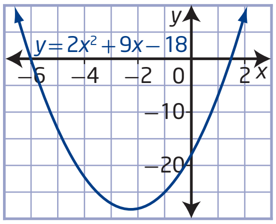
27. How many integers between 1 and 1000 do not contain repeated digits?

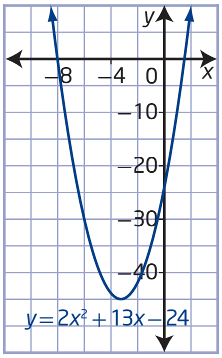
10. a) $y = |6 - x|$; domain $\{x \mid x \in R\}$, range $\{y \mid y \geq 0, y \in R\}$
b) $y = 4^x + 1$; domain $\{x \mid x \in R\}$, range $\{y \mid y \geq 1, y \in R\}$
c) $y = x^2$; domain $\{x \mid x \in R\}$, range $\{y \mid y \geq 0, y \in R\}$

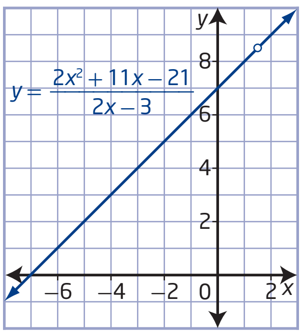
11. a) $r(x) = x - 200$; $t(x) = 0.72x$
b) $t(r(x)) = 0.72x - 144$; this represents applying federal taxes after deducting from her paycheque for her retirement.
c) $1800 **d)** $1744
e) The order changes the final amount. If you tax the income after subtracting $200, you are left with more money.

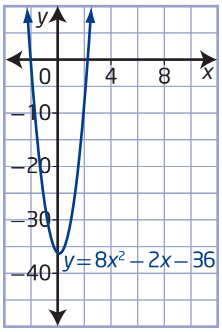
12. a)


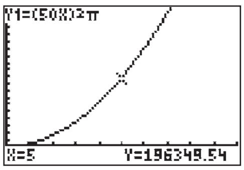
b) The function $f(t) = 10 \cos 2t$ is responsible for the periodic motion. The function $g(t) = 0.95^t$ is responsible for the exponential decay of the amplitude.

13. a) $y = 2x^2 + 9x - 18$


b) $y = 2x^2 + 13x - 24$


c) $y = x + 7, x \neq \frac{3}{2}$


d) $y = 8x^2 - 2x - 36$


14. a) $A(t) = 2500\pi t^2$
b)


c) approximately 196 350 cm^2
d) Example: No. In 30 s, the radius would be 1500 cm. Most likely the circular ripples would no longer be visible on the surface of the water due to turbulence.

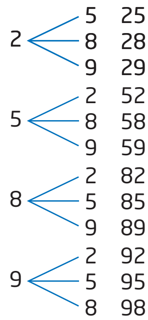
Chapter 11 Permutations, Combinations, and the Binomial Theorem

11.1 Permutations, pages 524 to 527

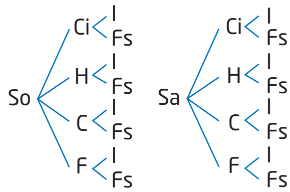
1. a)

Position 1	Position 2	Position 3
Jo	Amy	Mike
Jo	Mike	Amy
Amy	Jo	Mike
Amy	Mike	Jo
Mike	Jo	Amy
Mike	Amy	Jo

6 different arrangements

b)


12 different two-digit numbers

c) Use abbreviations: Soup (So), Salad (Sa), Chili (Ci), Hamburger (H), Chicken (C), Fish (F), Ice Cream (I) and Fruit Salad (Fs). 16 different meals


2. a) 56 **b)** 2520 **c)** 720 **d)** 4

3. Left Side $= 4! + 3!$
$= 4(3!) + 3!$
$= 5(3!)$

Right Side $= (4 + 3)!$
$= 7!$

Left Side $\neq$ Right Side

4. a) $9! = (9)(8)(7)(6)(5)(4)(3)(2)(1)$
$= 362\,880$
b) $\frac{9!}{5!4!} = \frac{(9)(8)(7)(6)(5!)}{(5!)(4)(3)(2)(1)}$
$= 126$
c) $(5!)(3!) = (5)(4)(3)(2)(1)(3)(2)(1)$
$= 720$
d) $6(4!) = 6(4)(3)(2)(1)$
$= 144$
e) $\frac{102!}{100!2!} = \frac{(102)(101)(100!)}{100!(2)(1)}$
$= (51)(101)$
$= 5151$
f) $7! - 5! = (7)(6)(5!) - 5!$
$= 41(5!)$
$= 4920$

5. a) 360 **b)** 420 **c)** 138 600
d) 20 **e)** 20 **f)** 10 080

6. 24 ways

7. a) $n = 6$ **b)** $n = 11$ **c)** $r = 2$
d) $n = 6$

8. a) 6 **b)** 35 **c)** 10

9. a) Case 1: first digit is 3 or 5; Case 2: first digit is 2 or 4
b) Case 1: first letter is a B; Case 2: first letter is an E

10. a) 48 **b)** 240 **c)** 48

11. a) 5040 **b)** 2520 **c)** 1440 **d)** 576

12. 720 total arrangements; 288 arrangements begin and end with a consonant.

13. No. The organization has 25 300 members but there are only 18 000 arrangements that begin with a letter other than O followed by three different digits.

14. 20

15. $266\frac{2}{3}$ h

16. a) 5040 **b)** 1440 **c)** 3600

17. a) 3360 **b)** 360

18. a) AABBS **b)** Example: TEETH

19. 3645 integers contain no 7s

20. a) 17 576 000

b) Example: Yes, Canada will eventually exceed 17.5 million postal communities.

21. a) 10^{14}

b) Yes, $10^{14} = 100\ 000\ 000\ 000\ 000$, which is 100 million million.

22. a) $r = 3$ **b)** $r = 7$ **c)** $n = 4$ **d)** $n = 42$

23. ${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$ and ${}_nP_n = n!$, so $0! = 1$.

24. The number of items to be arranged is less than the number of items in each set of arrangements.

25. 63 **26.** 84 **27.** 737 **28.** 15 **29.** 10

30. Example: Use the numbers 1 to 9 to represent the different students.

Day 1	Day 2	Day 3	Day 4
1 2 3	1 4 7	1 4 9	1 6 8
4 5 6	2 5 8	2 6 7	2 4 9
7 8 9	3 6 9	3 5 8	3 5 7

31. 24 zeros; Determine how many factors of 5 there are in 100!. Each multiple of 5 has one factor of 5 except 25, 50, 75, and 100, which have two factors of 5. So, there are 24 factors of 5 in 100!. There are more than enough factors of 2 to match up with the 5s to make factors of 10, so there are 24 zeros.

32. a) EDACB or BCADE **b)** 2

c) None. Since F only knows A, then F must stand next to A. However, in both arrangements from part a), A must stand between C and D, but F does not know either C or D and therefore cannot stand next to either of them. Therefore, no possible arrangement satisfies the conditions.

C1 a) ${}_aP_b = \frac{a!}{(a-b)!}$ is the formula for calculating the number of ways that b objects can be selected from a group of a objects, if order is important; for example, if you have a group of 20 students and you want to choose a team of 3 arranged from tallest to shortest.

b) $b \le a$

C2 By the fundamental counting principle, if the n objects are distinct, they can be arranged in $n!$ ways. However, if a of the objects are the same and b of the remaining objects are the same, then the number of different arrangements is reduced to $\frac{n!}{a!b!}$ to eliminate duplicates.

C3 a) $\frac{(n+2)(n+1)n}{4}$ **b)** $\frac{7+20r}{r(r+1)}$

C5 a) 362 880 **b)** 5.559 763… **c)** 6.559 763

d) Example: The answer to part c) is 1 more than the answer to part b). This is because $10! = 10(9!)$ and $\log 10! = \log 10 + \log 9! = 1 + \log 9!$.

11.2 Combinations, pages 534 to 536

1. a) Combination, because the order that you shake hands is not important.

b) Permutation, because the order of digits is important.

c) Combination, since the order that the cars are purchased is not important.

d) Combination, because the order that players are selected to ride in the van is not important.

2. ${}_5P_3$ is a permutation representing the number of ways of arranging 3 objects taken from a group of 5 objects. ${}_5C_3$ is a combination representing the number of ways of choosing any 3 objects from a group of 5 objects. ${}_5P_3 = 60$ and ${}_5C_3 = 10$.

3. a) ${}_6P_4 = 360$ **b)** ${}_7C_3 = 35$

c) ${}_5C_2 = 10$ **d)** ${}_{10}C_7 = 120$

4. a) 210 **b)** 5040

5. a) AB, AC, AD, BC, BD, CD

b) AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC

c) The number of permutations is 2! times the number of combinations.

6. a) $n = 10$ **b)** $n = 7$ **c)** $n = 4$ **d)** $n = 5$

7. a) Case 1: one-digit numbers, Case 2: two-digit numbers, Case 3: three-digit numbers

b) Cases of grouping the 4 members of the 5-member team from either grade: Case 1: four grade 12s, Case 2: three grade 12s and one grade 11, Case 3: two grade 12s and two grade 11s, Case 4: one grade 12 and three grade 11s, Case 5: four grade 11s

8. Left Side $= {}_{11}C_3$

$= \frac{11!}{(11-3)!3!}$

$= \frac{11!}{8!3!}$

Right Side $= {}_{11}C_8$

$= \frac{11!}{(11-8)!8!}$

$= \frac{11!}{3!8!}$

Left Side = Right Side

9. a) ${}_5C_5 = 1$

b) ${}_5C_0 = 1$; there is only one way to choose 5 objects from a group of 5 objects and only one way to choose 0 objects from a group of 5 objects.

10. a) 4 **b)** 10

11. a) 15 **b)** 22

12. Left Side

$= {}_nC_{r-1} + {}_nC_r$

$= \frac{n!}{(n-(r-1))!(r-1)!} + \frac{n!}{(n-r)!r!}$

$= \frac{n!}{(n-r+1)!(r-1)!} + \frac{n!}{(n-r)!r!}$

$= \frac{[n!(n-r)!r!] + [n!(n-r+1)!(r-1)!]}{(n-r+1)!(r-1)!(n-r)!r!}$

$= \frac{n!(n-r)!r(r-1)! + n!(n-r+1)(n-r)!(r-1)!}{(n-r+1)!(r-1)!(n-r)!r!}$

$= \frac{n!(n-r)!(r-1)![r + (n-r+1)]}{(n-r+1)!(r-1)!(n-r)!r!}$

$= \frac{n!\overset{1}{\cancel{(n-r)!}}\overset{1}{\cancel{(r-1)!}}(n+1)}{(n-r+1)!\underset{1}{\cancel{(r-1)!}}\underset{1}{\cancel{(n-r)!}}r!}$

$= \frac{n!(n+1)}{(n-r+1)!r!}$

$= \frac{(n+1)!}{(n-r+1)!r!}$

Right Side $= {}_{n+1}C_r$

$= \frac{(n+1)!}{(n+1-r)!r!}$

Left Side = Right Side

13. 20 different burgers; this is a combination because the order the ingredients is put on the burger is not important.