Your Turn

There is a logarithmic relationship between butterflies and flowers. In one study, scientists found that the relationship between the number, F, of flower species that a butterfly feeds on and the number, B, of butterflies observed can be modelled by the function $F = -2.641 + 8.958 \log B$.

Predict the number of butterfly observations in a region with 25 flower species.

Arctic butterfly, oeneis chryxus

Did You Know?

Eighty-seven different species of butterfly have been seen in Nunavut. Northern butterflies survive the winters in a larval stage and manufacture their own antifreeze to keep from freezing. They manage the cool summer temperatures by angling their wings to catch the sun's rays.

Key Ideas

- To represent real-life situations, you may need to transform the basic logarithmic function $y = \log_b x$ by applying reflections, stretches, and translations. These transformations should be performed in the same manner as those applied to any other function.
- The effects of the parameters a, b, h, and k in $y = a \log_c (b(x - h)) + k$ on the graph of the logarithmic function $y = \log_c x$ are shown below.

$$y = \underbrace{a}_{\text{Vertically stretch by a factor of } |a| \text{ about the } x\text{-axis. Reflect in the } x\text{-axis if } a < 0.} \log_c (\underbrace{b}_{\text{Horizontally stretch by a factor of } \left|\frac{1}{b}\right| \text{ about the } y\text{-axis. Reflect in the } y\text{-axis if } b < 0.}(x - \underbrace{h}_{\text{Horizontally translate } h \text{ units.}})) + \underbrace{k}_{\text{Vertically translate } k \text{ units.}}$$

- Only parameter h changes the vertical asymptote and the domain. None of the parameters change the range.

Check Your Understanding

Practise

1. Describe how the graph of each logarithmic function can be obtained from the graph of $y = \log_5 x$.
   a) $y = \log_5 (x - 1) + 6$
   b) $y = -4 \log_5 3x$
   c) $y = \frac{1}{2} \log_5 (-x) + 7$

2. a) Sketch the graph of $y = \log_3 x$, and then apply, in order, each of the following transformations.
   - Stretch vertically by a factor of 2 about the x-axis.
   - Translate 3 units to the left.

   b) Write the equation of the final transformed image.

3. a) Sketch the graph of $y = \log_2 x$, and then apply, in order, each of the following transformations.
   - Reflect in the y-axis.
   - Translate vertically 5 units up.

   b) Write the equation of the final transformed image.

4. Sketch the graph of each function.

   a) $y = \log_2 (x + 4) - 3$

   b) $y = -\log_3 (x + 1) + 2$

   c) $y = \log_4 (-2(x - 8))$

5. Identify the following characteristics of the graph of each function.

   i) the equation of the asymptote

   ii) the domain and range

   iii) the y-intercept, to one decimal place if necessary

   iv) the x-intercept, to one decimal place if necessary

   a) $y = -5 \log_3 (x + 3)$

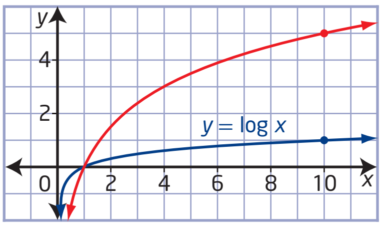
   b) $y = \log_6 (4(x + 9))$

   c) $y = \log_5 (x + 3) - 2$

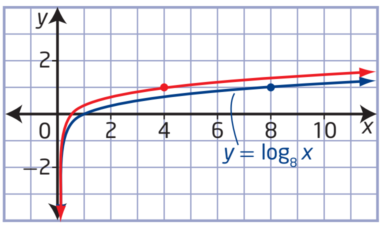
   d) $y = -3 \log_2 (x + 1) - 6$

6. In each, the red graph is a stretch of the blue graph. Write the equation of each red graph.

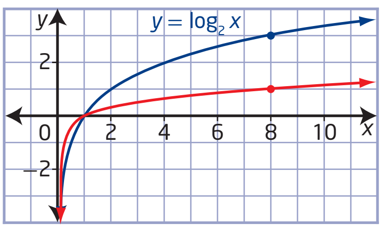
   a)



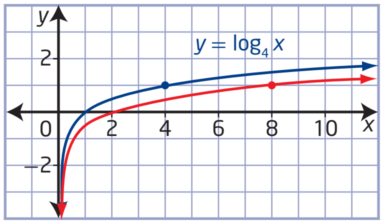
   b)



   c)



   d)



7. Describe, in order, a series of transformations that could be applied to the graph of $y = \log_7 x$ to obtain the graph of each function.

   a) $y = \log_7 (4(x + 5)) + 6$

   b) $y = 2 \log_7 \left(-\frac{1}{3}(x - 1)\right) - 4$

Apply

8. The graph of $y = \log_3 x$ has been transformed to $y = a \log_3 (b(x - h)) + k$. Find the values of a, b, h, and k for each set of transformations. Write the equation of the transformed function.

   a) a reflection in the x-axis and a translation of 6 units left and 3 units up

   b) a vertical stretch by a factor of 5 about the x-axis and a horizontal stretch about the y-axis by a factor of $\frac{1}{3}$

   c) a vertical stretch about the x-axis by a factor of $\frac{3}{4}$, a horizontal stretch about the y-axis by a factor of 4, a reflection in the y-axis, and a translation of 2 units right and 5 units down

9. Describe how the graph of each logarithmic function could be obtained from the graph of $y = \log_3 x$.

   a) $y = 5 \log_3 (-4x + 12) - 2$

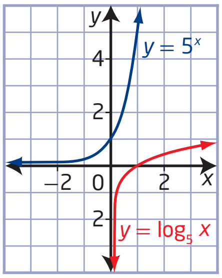
   b) $y = -\frac{1}{4} \log_3 (6 - x) + 1$

10. a) Only a vertical translation has been applied to the graph of $y = \log_3 x$ so that the graph of the transformed image passes through the point $(9, -4)$. Determine the equation of the transformed image.

    b) Only a horizontal stretch has been applied to the graph of $y = \log_2 x$ so that the graph of the transformed image passes through the point $(8, 1)$. Determine the equation of the transformed image.

8. a) $y = \log_5 x$

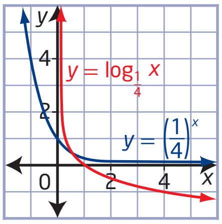
b)



domain $\{x \mid x > 0, x \in R\}$,
range $\{y \mid y \in R\}$,
x-intercept 1,
no y-intercept,
vertical asymptote $x = 0$

9. a) $g^{-1}(x) = \left(\frac{1}{4}\right)^x$

b)



domain $\{x \mid x \in R\}$,
range $\{y \mid y > 0, y \in R\}$,
no x-intercept,
y-intercept 1,
horizontal asymptote $y = 0$

10. They are reflections of each other in the line $y = x$.

11. a) They have the exact same shape.

b) One of them is increasing and the other is decreasing.

12. a) 216 **b)** 81 **c)** 64 **d)** 8

13. a) 7 **b)** 6

14. a) 0 **b)** 1

15. -1

16. 16

17. a) $t = \log_{1.1} N$ **b)** 145 days

18. The larger asteroid had a relative risk that was 1479 times as dangerous.

19. 1000 times as great

20. 5

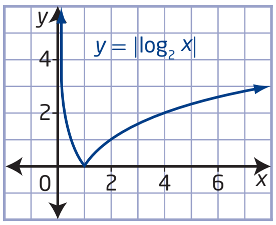
21. $m = 14$, $n = 13$

22. $4n$

23. $y = 3^{2^x}$

24. $n = 8$; $m = 3$

C1



The function has the same general shape, but instead of decreasing, after $x = 1$ the function increases without limit.

C2 Answers will vary.

C3 **Step 1: a)** $e = 2.718\ 281\ 828$ **b)** 10^{10}

Step 2: a) domain $\{x \mid x > 0, x \in R\}$, range $\{y \mid y \in R\}$, x-intercept 1, no y-intercept, vertical asymptote $x = 0$

b) $y = \ln x$

Step 3: a) $r = 2.41$

b) i) $\theta = \dfrac{\ln r}{0.14}$ **ii)** $\theta = 17.75$

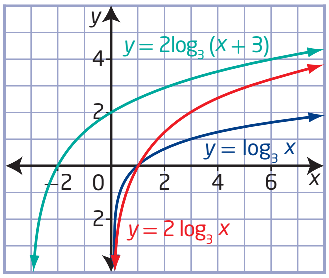
8.2 Transformations of Logarithmic Functions, pages 389 to 391

1. a) Translate 1 unit right and 6 units up.

b) Reflect in the x-axis, stretch vertically about the x-axis by a factor of 4, and stretch horizontally about the y-axis by a factor of $\frac{1}{3}$.

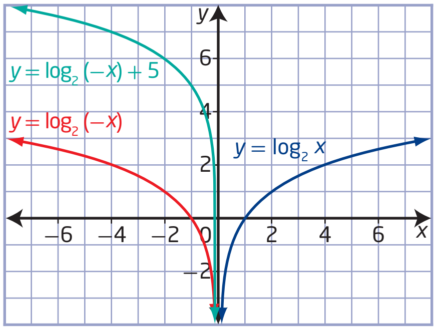
c) Reflect in the y-axis, stretch vertically about the x-axis by a factor of $\frac{1}{2}$, and translate 7 units up.

2. a)



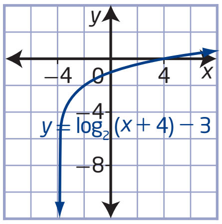
b) $y = 2 \log_3 (x + 3)$

3. a)

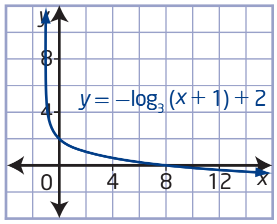


b) $y = \log_2 (-x) + 5$

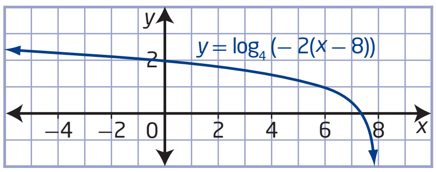
4. a)



b)



c)



5. a) **i)** vertical asymptote $x = -3$
ii) domain $\{x \mid x > -3, x \in R\}$, range $\{y \mid y \in R\}$
iii) y-intercept -5 **iv)** x-intercept -2

b) **i)** vertical asymptote $x = -9$
ii) domain $\{x \mid x > -9, x \in R\}$, range $\{y \mid y \in R\}$
iii) y-intercept 2 **iv)** x-intercept -8.75

c) **i)** vertical asymptote $x = -3$
ii) domain $\{x \mid x > -3, x \in R\}$, range $\{y \mid y \in R\}$
iii) y-intercept -1.3 **iv)** x-intercept 22

d) **i)** vertical asymptote $x = -1$
ii) domain $\{x \mid x > -1, x \in R\}$, range $\{y \mid y \in R\}$
iii) y-intercept -6 **iv)** x-intercept $-\frac{3}{4}$

6. a) $y = 5 \log x$ b) $y = \log_8 2x$
c) $y = \frac{1}{3} \log_2 x$ d) $y = \log_4 \left(\frac{x}{2}\right)$

7. a) stretch horizontally about the y-axis by a factor of $\frac{1}{4}$; translate 5 units left and 6 units up
b) stretch horizontally about the y-axis by a factor of 3; stretch vertically about the x-axis by a factor of 2; reflect in the y-axis; translate 1 unit right and 4 units down

8. a) $a = -1$, $b = 1$, $h = -6$, $k = 3$; $y = -\log_3 (x + 6) + 3$
b) $a = 5$, $b = 3$, $h = 0$, $k = 0$; $y = 5 \log_3 3x$
c) $a = 0.75$, $b = -0.25$, $h = 2$, $k = -5$;
$y = \frac{3}{4} \log_3 \left(-\frac{1}{4}(x - 2)\right) - 5$

9. a) Reflect in the y-axis, stretch vertically about the x-axis by a factor of 5, stretch horizontally about the y-axis by a factor of $\frac{1}{4}$, and translate 3 units right and 2 units down.
b) Reflect in the x-axis, reflect in the y-axis, stretch vertically about the x-axis by a factor of $\frac{1}{4}$, translate 6 units right and 1 unit up.

10. a) $y = \log_3 x - 6$ b) $y = \log_2 \left(\frac{x}{4}\right)$

11. Stretch vertically about the x-axis by a factor of 3 and translate 4 units right and 2 units down.

12. a) Stretch vertically about the x-axis by a factor of 0.67, stretch horizontally about the y-axis by a factor of $\frac{25}{9}$ or approximately 2.78, and translate 1.46 units up.
b) 515 649 043 kWh

13. a) 0.8 µL b) 78 mmHg

14. a) 172 cm b) 40 kg

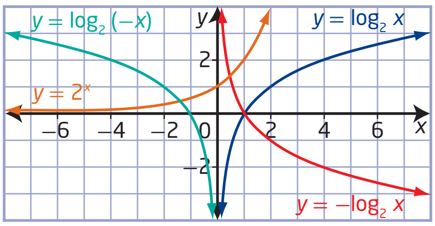
15. $a = \frac{1}{3}$

16. a) $y = -2 \log_5 x + 13$ b) $y = \log 2x$

17. $a = \frac{1}{2}$, $k = -8$

C1 $a = \frac{1}{4}$, $b = \frac{1}{3}$, $h = 4$, $k = -1$;
$g(x) = 0.25 \log_5 \left(\frac{1}{3}\right)(x - 4) - 1$

C2 a) $y = -\log_2 x$, $y = \log_2 (-x)$, $y = 2^x$
b) Reflect in the x-axis, reflect in the y-axis, and reflect in the line $y = x$.



C3 a) $y = \frac{1}{2} \log_7 \frac{(x - 5)}{3} + \frac{1}{2}$ b) $y = 3^{\frac{x-8}{2}} + 1$

C4 Answers will vary.

8.3 Laws of Logarithms, pages 400 to 403

1. a) $\log_7 x + 3 \log_7 y + \frac{1}{2} \log_7 z$
b) $8(\log_5 x + \log_5 y + \log_5 z)$
c) $2 \log x - \log y - \frac{1}{3} \log z$
d) $y = \log_3 x + \left(\frac{1}{2}\right)(\log_3 y - \log_3 z)$

2. a) 2 b) 3 c) 3.5 d) 3

3. a) $\log_9 \left(\frac{xz^4}{y}\right)$ b) $y = \log_3 \frac{\sqrt{x}}{y^2}$
c) $\log_6 \left(\frac{x}{\sqrt[5]{xy^2}}\right)$ d) $\log \sqrt[3]{xy}$

4. a) 1.728 b) 1.44 c) 1.2

5. a) 27 b) 49

6. a) Stretch horizontally about the y-axis by a factor of $\frac{1}{8}$.
b) Translate 3 units up.

7. a) False; the division must take place inside the logarithm.
b) False; it must be a multiplication inside the logarithm.
c) True
d) False; the power must be inside the logarithm.
e) True

8. a) $P - Q$ b) $P + Q$ c) $P + \frac{Q}{2}$ d) $2Q - 2P$

9. a) $6K$ b) $1 + K$ c) $2K + 2$ d) $\frac{K}{5} - 3$

10. a) $\frac{1}{2} \log_5 x$, $x > 0$ b) $\frac{2}{3} \log_{11} x$, $x > 0$

11. a) $\log_2 \left(\frac{x + 5}{3}\right)$, $x < -5$ or $x > 5$
b) $\log_7 \left(\frac{x + 4}{x + 2}\right)$, $x < -4$ or $x > 4$
c) $\log_8 \left(\frac{x + 3}{x - 2}\right)$, $x > 2$

12. a) Left Side $= \log_c 48 - (\log_c 3 + \log_c 2)$
$= \log_c 48 - \log_c 6$
$= \log_c 8$
= Right Side
b) Left Side $= 7 \log_c 4$
$= 7 \log_c 2^2$
$= 2(7) \log_c 2$
$= 14 \log_c 2$
= Right Side
c) Left Side $= \frac{1}{2}(\log_c 2 + \log_c 6)$
$= \frac{1}{2}(\log_c 2 + \log_c 3 + \log_c 2)$
$= \frac{1}{2}(2 \log_c 2) + \frac{1}{2} \log_c 3$
$= \log_c 2 + \log_c \sqrt{3}$
= Right Side
d) Left Side $= \log_c (5c)^2$
$= 2 \log_c 5c$
$= 2 (\log_c 5 + \log_c c)$
$= 2 (\log_c 5 + 1)$
= Right Side

13. a) 70 dB b) approximately 1995 times as loud
c) approximately 98 dB

14. Decibels must be changed to intensity to gauge loudness. The function that maps the change is not linear.

15. 3.2 V

16. a) 10^{-7} mol/L b) 12.6 times as acidic c) 3.4

17. 0.18 km/s

18. a) The graphs are the same for $x > 0$. However, the graph of $y = \log x^2$ has a second branch for $x < 0$, which is the reflection in the y-axis of the branch for $x > 0$.
b) The domains are different. The function $y = \log x^2$ is defined for all values of x except 0, while the function $y = 2 \log x$ is defined only for $x > 0$.
c) $x > 0$