Key Ideas

- To solve a trigonometric equation algebraically, you can use the same techniques as used in solving linear and quadratic equations.
- When you arrive at $\sin \theta = a$ or $\cos \theta = a$ or $\tan \theta = a$, where $a \in R$, then use the unit circle for exact values of θ and inverse trigonometric function keys on a calculator for approximate measures. Use reference angles to find solutions in other quadrants.
- To solve a trigonometric equation involving $\csc \theta$, $\sec \theta$, or $\cot \theta$, you may need to work with the related reciprocal value(s).
- To determine a general solution or if the domain is real numbers, find the solutions in one positive rotation (2π or 360°). Then, use the concept of coterminal angles to write an expression that identifies all possible measures.

Check Your Understanding

Practise

1. Without solving, determine the number of solutions for each trigonometric equation in the specified domain. Explain your reasoning.

a) $\sin \theta = \frac{\sqrt{3}}{2}$, $0 \leq \theta < 2\pi$

b) $\cos \theta = \frac{1}{\sqrt{2}}$, $-2\pi \leq \theta < 2\pi$

c) $\tan \theta = -1$, $-360° \leq \theta \leq 180°$

d) $\sec \theta = \frac{2\sqrt{3}}{3}$, $-180° \leq \theta < 180°$

2. The equation $\cos \theta = \frac{1}{2}$, $0 \leq \theta < 2\pi$, has solutions $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. Suppose the domain is not restricted.

a) What is the general solution corresponding to $\theta = \frac{\pi}{3}$?

b) What is the general solution corresponding to $\theta = \frac{5\pi}{3}$?

3. Determine the exact roots for each trigonometric equation or statement in the specified domain.

a) $2 \cos \theta - \sqrt{3} = 0$, $0 \leq \theta < 2\pi$

b) $\csc \theta$ is undefined, $0° \leq \theta < 360°$

c) $5 - \tan^2 \theta = 4$, $-180° \leq \theta \leq 360°$

d) $\sec \theta + \sqrt{2} = 0$, $-\pi \leq \theta \leq \frac{3\pi}{2}$

4. Solve each equation for $0 \leq \theta < 2\pi$. Give solutions to the nearest hundredth of a radian.

a) $\tan \theta = 4.36$

b) $\cos \theta = -0.19$

c) $\sin \theta = 0.91$

d) $\cot \theta = 12.3$

e) $\sec \theta = 2.77$

f) $\csc \theta = -1.57$

5. Solve each equation in the specified domain.

a) $3 \cos \theta - 1 = 4 \cos \theta$, $0 \leq \theta < 2\pi$

b) $\sqrt{3} \tan \theta + 1 = 0$, $-\pi \leq \theta \leq 2\pi$

c) $\sqrt{2} \sin x - 1 = 0$, $-360° < x \leq 360°$

d) $3 \sin x - 5 = 5 \sin x - 4$, $-360° \leq x < 180°$

e) $3 \cot x + 1 = 2 + 4 \cot x$, $-180° < x < 360°$

f) $\sqrt{3} \sec \theta + 2 = 0$, $-\pi \leq \theta \leq 3\pi$

6. Copy and complete the table to express each domain or interval using the other notation.

	Domain	Interval Notation
a)	$-2\pi \le \theta \le 2\pi$	
b)	$-\frac{\pi}{3} \le \theta \le \frac{7\pi}{3}$	
c)	$0° \le \theta \le 270°$	
d)		$\theta \in [0, \pi)$
e)		$\theta \in (0°, 450°)$
f)		$\theta \in (-2\pi, 4\pi]$

7. Solve for θ in the specified domain. Give solutions as exact values where possible. Otherwise, give approximate measures to the nearest thousandth.

a) $2 \cos^2 \theta - 3 \cos \theta + 1 = 0, 0 \le \theta < 2\pi$

b) $\tan^2 \theta - \tan \theta - 2 = 0, 0° \le \theta < 360°$

c) $\sin^2 \theta - \sin \theta = 0, \theta \in [0, 2\pi)$

d) $\sec^2 \theta - 2 \sec \theta - 3 = 0,$
$\theta \in [-180°, 180°)$

8. Todd believes that 180° and 270° are solutions to the equation $5 \cos^2 \theta = -4 \cos \theta$. Show how you would check to determine whether Todd's solutions are correct.

Apply

9. Aslan and Shelley are finding the solution for $2 \sin^2 \theta = \sin \theta, 0 < \theta \le \pi$. Here is their work.

$2\sin^2 \theta = \sin \theta$

$\frac{2\sin^2 \theta}{\sin \theta} = \frac{\sin \theta}{\sin \theta}$ Step 1

$2\sin \theta = 1$ Step 2

$\sin \theta = \frac{1}{2}$ Step 3

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ Step 4

a) Identify the error that Aslan and Shelley made and explain why their solution is incorrect.

b) Show a correct method to determine the solution for $2 \sin^2 \theta = \sin \theta, 0 < \theta \le \pi$.

10. Explain why the equation $\sin \theta = 0$ has no solution in the interval $(\pi, 2\pi)$.

11. What is the solution for $\sin \theta = 2$? Show how you know. Does the interval matter?

12. Jaycee says that the trigonometric equation $\cos \theta = \frac{1}{2}$ has an infinite number of solutions. Do you agree? Explain.

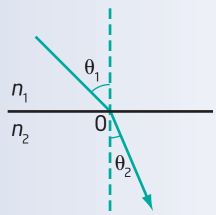
13. a) Helene is asked to solve the equation $3 \sin^2 \theta - 2 \sin \theta = 0, 0 \le \theta \le \pi$. She finds that $\theta = \pi$. Show how she could check whether this is a correct root for the equation.

b) Find all the roots of the equation $3 \sin^2 \theta - 2 \sin \theta = 0, \theta \in [0, \pi]$.

14. Refer to the Did You Know? below. Use Snell's law of refraction to determine the angle of refraction of a ray of light passing from air into water if the angle of incidence is 35°. The refractive index is 1.000 29 for air and 1.33 for water.

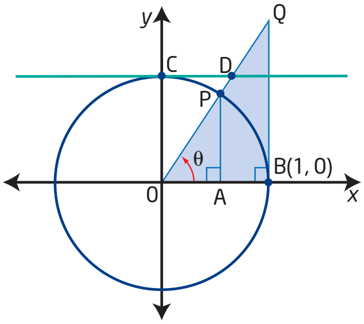
Did You Know?

Willebrord Snell, a Dutch physicist, discovered that light is bent (refracted) as it passes from one medium into another. Snell's law is shown in the diagram.



$n_1 \sin \theta_1 = n_2 \sin \theta_2$,
where θ_1 is the angle of incidence,
θ_2 is the angle of refraction, and
n_1 and n_2 are the refractive indices of the mediums.

b)



In △OCD, ∠ODC = θ (alternate angles). Then, $\sin \theta = \frac{OC}{OD} = \frac{1}{OD}$. So, $\csc \theta = \frac{1}{\sin \theta} = OD$.
Similarly, $\cot \theta = CD$.

C1 a) Paula is correct. Examples: $\sin 0° = 0$, $\sin 10° \approx 0.1736$, $\sin 25° \approx 0.4226$, $\sin 30° = 0.5$, $\sin 45° \approx 0.7071$, $\sin 60° \approx 0.8660$, $\sin 90° = 1$.

b) In quadrant II, sine decreases from $\sin 90° = 1$ to $\sin 180° = 0$. This happens because the y-value of points on the unit circle are decreasing toward the horizontal axis as the value of the angle moves from 90° to 180°.

c) Yes, the sine ratio increases in quadrant IV, from its minimum value of −1 at 270° up to 0 at 0°.

C2 When you draw its diagonals, the hexagon is composed of six equilateral triangles. On the diagram shown, each vertex will be 60° from the previous one. So, the coordinates, going in a positive direction from (1, 0) are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $(-1, 0)$, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

C3 a) $\text{slope}_{OP} = \frac{\sin \theta}{\cos \theta}$ or $\tan \theta$

b) Yes, this formula applies in each quadrant. In quadrant II, sin θ is negative, which makes the slope negative, as expected. Similar reasoning applies in the other quadrants.

c) $y = \left(\frac{\sin \theta}{\cos \theta}\right)x$ or $y = (\tan \theta)x$

d) Any line whose slope is defined can be translated vertically by adding the value of the y-intercept b. The equation will be $y = \left(\frac{\sin \theta}{\cos \theta}\right)x + b$ or $y = (\tan \theta)x + b$.

C4 a) $\frac{4}{5}$ **b)** $\frac{3}{5}$ **c)** $\frac{5}{4}$ **d)** $-\frac{4}{5}$

4.4 Introduction to Trigonometric Equations, pages 211 to 214

1. a) two solutions; sin θ is positive in quadrants I and II

b) four solutions; cos θ is positive in quadrants I and IV, giving two solutions for each of the two complete rotations

c) three solutions; tan θ is negative in quadrants II and IV, and the angle rotates through these quadrants three times from −360° to 180°

d) two solutions; sec θ is positive in quadrants I and IV and the angle is in each quadrant once from −180° to 180°

2. a) $\theta = \frac{\pi}{3} + 2\pi n, n \in I$ **b)** $\theta = \frac{5\pi}{3} + 2\pi n, n \in I$

3. a) $\theta = \frac{\pi}{6}, \frac{11\pi}{6}$ **b)** $\theta = 0°, 180°$

c) $\theta = -135°, -45°, 45°, 135°, 225°, 315°$

d) $\theta = -\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$

4. a) $\theta = 1.35, 4.49$ **b)** $\theta = 1.76, 4.52$

c) $\theta = 1.14, 2.00$ **d)** $\theta = 0.08, 3.22$

e) 1.20 and 5.08 **f)** 3.83 and 5.59

5. a) $\theta = \pi$ **b)** $\theta = -\frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}$

c) $x = -315°, -225°, 45°, 135°$

d) $x = -150°, -30°$

e) $x = -45°, 135°, 315°$

f) $\theta = -\frac{5\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}$

6. a) $\theta \in [-2\pi, 2\pi]$ **b)** $\theta \in \left[-\frac{\pi}{3}, \frac{7\pi}{3}\right]$

c) $\theta \in [0°, 270°]$ **d)** $0 \le \theta < \pi$

e) $0° < \theta < 450°$ **f)** $-2\pi < \theta \le 4\pi$

7. a) $\theta = 0, \frac{\pi}{3}, \frac{5\pi}{3}$

b) $\theta = 63.435°, 243.435°, 135°, 315°$

c) $\theta = 0, \frac{\pi}{2}, \pi$

d) $\theta = -180°, -70.529°, 70.529°$

8. Check for θ = 180°.
Left Side = $5(\cos 180°)^2 = 5(-1)^2 = 5$
Right Side = $-4 \cos 180° = -4(-1) = 4$
Since Left Side ≠ Right Side, θ = 180° is not a solution.
Check for θ = 270°.
Left Side = $5(\cos 270°)^2 = 5(0)^2 = 0$
Right Side = $-4 \cos 270° = -4(0) = 0$
Since Left Side = Right Side, θ = 270° is a solution.

9. a) They should not have divided both sides of the equation by sin θ. This will eliminate one of the possible solutions.

b)
$$2 \sin^2 \theta = \sin \theta$$
$$2 \sin^2 \theta - \sin \theta = 0$$
$$\sin \theta(2 \sin \theta - 1) = 0$$
$$\sin \theta = 0 \quad \text{and} \quad 2 \sin \theta - 1 = 0$$
$$\sin \theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \pi$$

10. Sin θ = 0 when θ = 0, π, and 2π but none of these values are in the interval (π, 2π).

11. Sin θ is only defined for the values $-1 \le \sin \theta \le 1$, and 2 is outside this range, so sin θ = 2 has no solution.

12. Yes, the general solutions are $\theta = \frac{\pi}{3} + 2\pi n, n \in I$ and $\theta = \frac{5\pi}{3} + 2\pi n, n \in I$. Since there are an infinite number of integers, there will be an infinite number of solutions coterminal with $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

13. a) Helene can check her work by substituting π for θ in the original equation.
Left Side = $3(\sin \pi)^2 - 2 \sin \pi$
= $3(0)^2 - 2(0)$
= 0
= Right Side

b) $\theta = 0, 0.7297, 2.4119, \pi$

14. 25.56°

15. a) June **b)** December

c) Yes. Greatest sales of air conditioners be expected to happen before the hottest months (June) and the least sales before the coldest months (December).