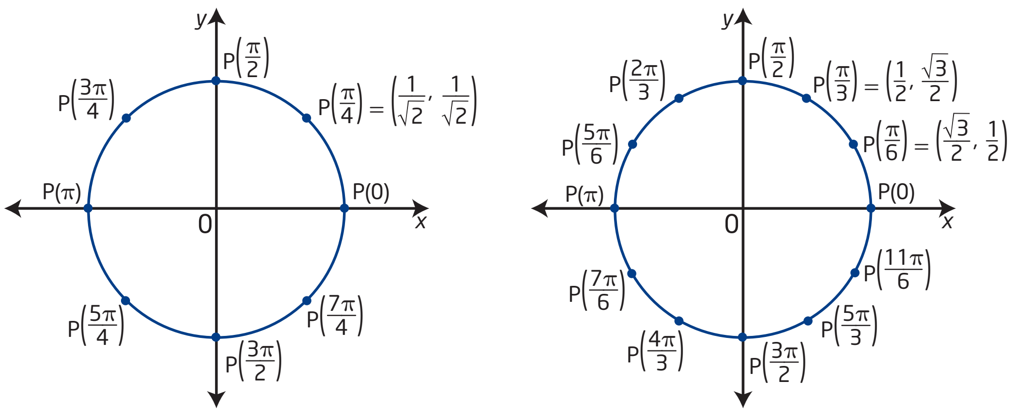
Key Ideas

- The equation for the unit circle is $x^2 + y^2 = 1$. It can be used to determine whether a point is on the unit circle or to determine the value of one coordinate given the other. The equation for a circle with centre at (0, 0) and radius r is $x^2 + y^2 = r^2$.
- On the unit circle, the measure in radians of the central angle and the arc subtended by that central angle are numerically equivalent.
- Some of the points on the unit circle correspond to exact values of the special angles learned previously.
- You can use patterns to determine coordinates of points. For example, the numerical value of the coordinates of points on the unit circle change to their opposite sign every $\frac{1}{2}$ rotation.

  If $P(\theta) = (a, b)$ is in quadrant I, then both a and b are positive. $P(\theta + \pi)$ is in quadrant III. Its coordinates are $(-a, -b)$, where $a > 0$ and $b > 0$.



Check Your Understanding

Practise

1. Determine the equation of a circle with centre at the origin and radius
   a) 4 units
   b) 3 units
   c) 12 units
   d) 2.6 units

2. Is each point on the unit circle? How do you know?
   a) $\left(-\frac{3}{4}, \frac{1}{4}\right)$
   b) $\left(\frac{\sqrt{5}}{8}, \frac{7}{8}\right)$
   c) $\left(-\frac{5}{13}, \frac{12}{13}\right)$
   d) $\left(\frac{4}{5}, -\frac{3}{5}\right)$
   e) $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
   f) $\left(\frac{\sqrt{7}}{4}, \frac{3}{4}\right)$

3. Determine the missing coordinate(s) for all points on the unit circle satisfying the given conditions. Draw a diagram to support your answer.

a) $\left(\frac{1}{4}, y\right)$ in quadrant I

b) $\left(x, \frac{2}{3}\right)$ in quadrant II

c) $\left(-\frac{7}{8}, y\right)$ in quadrant III

d) $\left(x, -\frac{5}{7}\right)$ in quadrant IV

e) $\left(x, \frac{1}{3}\right)$, where $x < 0$

f) $\left(\frac{12}{13}, y\right)$, not in quadrant I

4. If P(θ) is the point at the intersection of the terminal arm of angle θ and the unit circle, determine the exact coordinates of each of the following.

a) $P(\pi)$

b) $P\left(-\frac{\pi}{2}\right)$

c) $P\left(\frac{\pi}{3}\right)$

d) $P\left(-\frac{\pi}{6}\right)$

e) $P\left(\frac{3\pi}{4}\right)$

f) $P\left(-\frac{7\pi}{4}\right)$

g) $P(4\pi)$

h) $P\left(\frac{5\pi}{2}\right)$

i) $P\left(\frac{5\pi}{6}\right)$

j) $P\left(-\frac{4\pi}{3}\right)$

5. Identify a measure for the central angle θ in the interval $0 \leq \theta < 2\pi$ such that P(θ) is the given point.

a) $(0, -1)$

b) $(1, 0)$

c) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

d) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

e) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

f) $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

g) $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

h) $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

i) $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

j) $(-1, 0)$

6. Determine one positive and one negative measure for θ if $P(\theta) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Apply

7. Draw a diagram of the unit circle.

a) Mark two points, P(θ) and P(θ + π), on your diagram. Use measurements to show that these points have the same coordinates except for their signs.

b) Choose a different quadrant for the original point, P(θ). Mark it and P(θ + π) on your diagram. Is the result from part a) still true?

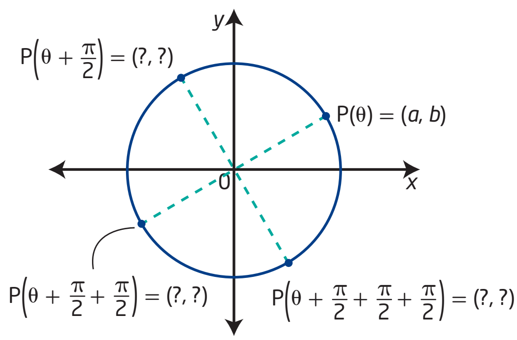
8. **MINI LAB** Determine the pattern in the coordinates of points that are $\frac{1}{4}$ rotation apart on the unit circle.

Step 1 Start with the points $P(0) = (1, 0)$, $P\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and $P\left(\frac{5\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Show these points on a diagram.

Step 2 Move $+\frac{1}{4}$ rotation from each point. Determine each new point and its coordinates. Show these points on your diagram from step 1.

Step 3 Move $-\frac{1}{4}$ rotation from each original point. Determine each new point and its coordinates. Mark these points on your diagram.

Step 4 How do the values of the *x*-coordinates and *y*-coordinates of points change with each quarter-rotation? Make a copy of the diagram and complete the coordinates to summarize your findings.



9. Use the diagram below to help answer these questions.

a) What is the equation of this circle?

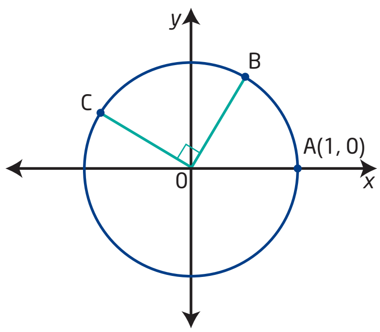
b) If the coordinates of C are $\left(-\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$, what are the coordinates of B?

c) If the measure of $\overset{\frown}{AB}$ is θ, what is an expression for the measure of $\overset{\frown}{AC}$?

Note: $\overset{\frown}{AB}$ means the arc length from A to B.

d) Let P(θ) = B. In which quadrant is $P\left(\theta - \frac{\pi}{2}\right)$?

e) What are the maximum and minimum values for either the *x*-coordinates or *y*-coordinates of points on the unit circle?



10. Mya claims that every value of *x* between 0 and 1 can be used to find the coordinates of a point on the unit circle in quadrant I.

a) Do you agree with Mya? Explain.

b) Mya showed the following work to find the *y*-coordinate when *x* = 0.807.

$y = 1 - (0.807)^2$
$= 0.348\ 751$

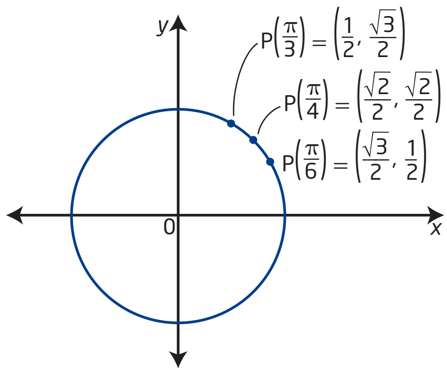
The point on the unit circle is (0.807, 0.348 751).

How can you check Mya's answer? Is she correct? If not, what is the correct answer?

c) If *y* = 0.2571, determine *x* so the point is on the unit circle and in the first quadrant.

11. Wesley enjoys tricks and puzzles. One of his favourite tricks involves remembering the coordinates for $P\left(\frac{\pi}{3}\right)$, $P\left(\frac{\pi}{4}\right)$, and $P\left(\frac{\pi}{6}\right)$. He will not tell you his trick. However, you can discover it for yourself.

a) Examine the coordinates shown on the diagram.



b) What do you notice about the denominators?

c) What do you notice about the numerators of the *x*-coordinates? Compare them with the numerators of the *y*-coordinates. Why do these patterns make sense?

d) Why are square roots involved?

e) Explain this memory trick to a partner.

12. a) Explain, with reference to the unit circle, what the interval $-2\pi \le \theta < 4\pi$ represents.

b) Use your explanation to determine all values for θ in the interval $-2\pi \le \theta < 4\pi$ such that $P(\theta) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

c) How do your answers relate to the word "coterminal"?

13. If $P(\theta) = \left(-\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right)$, determine the following.

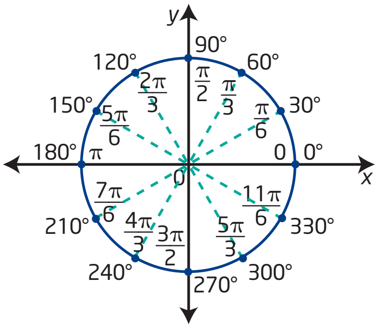
a) What does P(θ) represent? Explain using a diagram.

b) In which quadrant does θ terminate?

c) Determine the coordinates of $P\left(\theta + \frac{\pi}{2}\right)$.

d) Determine the coordinates of $P\left(\theta - \frac{\pi}{2}\right)$.

b)



C5 a) $x = 3$ **b)** $y = x - 3$

4.2 The Unit Circle, pages 186 to 190

1. a) $x^2 + y^2 = 16$ **b)** $x^2 + y^2 = 9$
c) $x^2 + y^2 = 144$ **d)** $x^2 + y^2 = 6.76$

2. a) No; $\left(-\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{5}{8} \neq 1$

b) No; $\left(\frac{\sqrt{5}}{8}\right)^2 + \left(\frac{7}{8}\right)^2 = \frac{27}{32} \neq 1$

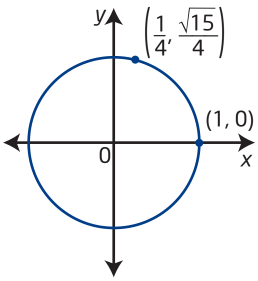
c) Yes; $\left(-\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = 1$

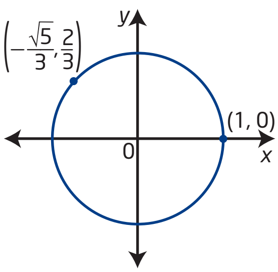
d) Yes; $\left(\frac{4}{5}\right)^2 + \left(-\frac{3}{5}\right)^2 = 1$

e) Yes; $\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 = 1$

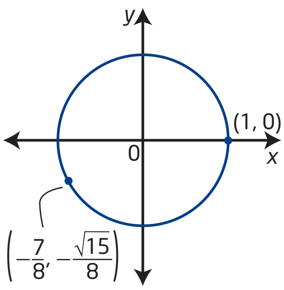
f) Yes; $\left(\frac{\sqrt{7}}{4}\right)^2 + \left(\frac{3}{4}\right)^2 = 1$

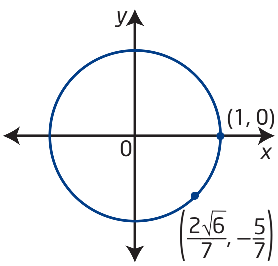
3. a) $y = \frac{\sqrt{15}}{4}$ **b)** $x = -\frac{\sqrt{5}}{3}$



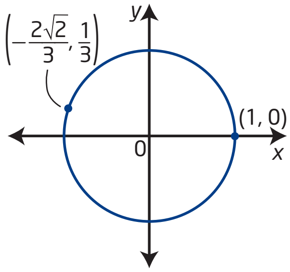


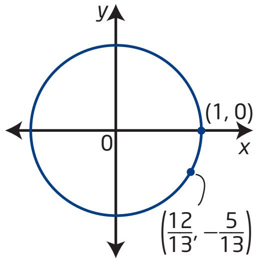
c) $y = -\frac{\sqrt{15}}{8}$ **d)** $x = \frac{2\sqrt{6}}{7}$





e) $x = -\frac{2\sqrt{2}}{3}$ **f)** $y = -\frac{5}{13}$



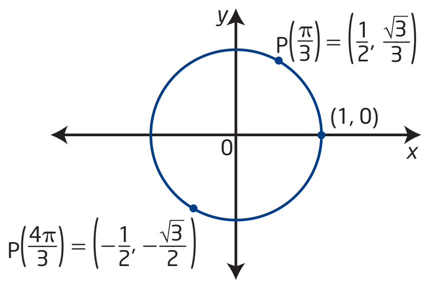


4. a) $(-1, 0)$ **b)** $(0, -1)$
c) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **d)** $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
e) $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **f)** $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
g) $(1, 0)$ **h)** $(0, 1)$
i) $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **j)** $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

5. a) $\frac{3\pi}{2}$ **b)** 0 **c)** $\frac{\pi}{4}$ **d)** $\frac{3\pi}{4}$
e) $\frac{\pi}{3}$ **f)** $\frac{5\pi}{3}$ **g)** $\frac{5\pi}{6}$ **h)** $\frac{7\pi}{6}$
i) $\frac{5\pi}{4}$ **j)** π

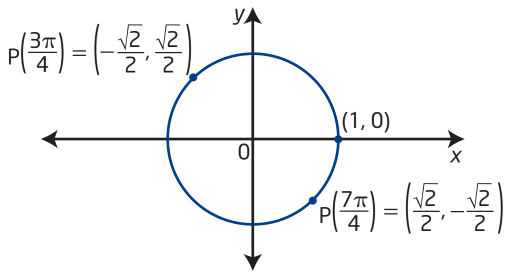
6. $\frac{5\pi}{6}$ and $-\frac{7\pi}{6}$

7. a)



If $\theta = \frac{\pi}{3}$ then $\theta + \pi = \frac{\pi}{3} + \pi$ or $\frac{4\pi}{3}$ since $P\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $P\left(\frac{4\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

b)



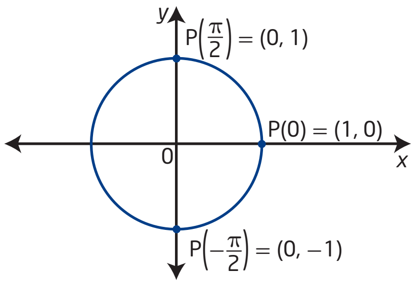
If $\theta = \frac{3\pi}{4}$ then $\theta + \pi = \frac{3\pi}{4} + \pi$ or $\frac{7\pi}{4}$ since $P\left(\frac{3\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $P\left(\frac{7\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

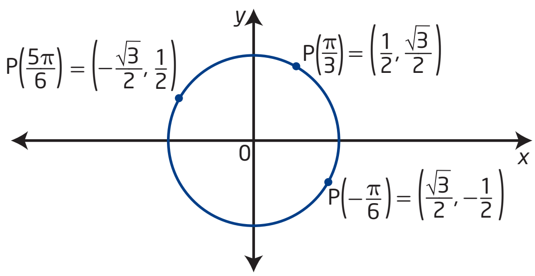
8.

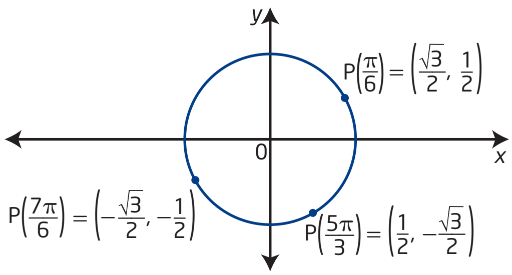
Point	$+\frac{1}{4}$ rotation	$-\frac{1}{4}$ rotation	Step 4: Description
$P(0) = (1, 0)$	$P\left(\frac{\pi}{2}\right) = (0, 1)$	$P\left(-\frac{\pi}{2}\right) = (0, -1)$	*x*- and *y*-values change places and take signs of new quadrant
$P\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$P\left(\frac{\pi}{3} + \frac{\pi}{2}\right) = P\left(\frac{5\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$P\left(\frac{\pi}{3} - \frac{\pi}{2}\right) = P\left(-\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	*x*- and *y*-values change places and take signs of new quadrant
$P\left(\frac{5\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$P\left(\frac{5\pi}{3} + \frac{\pi}{2}\right) = P\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$P\left(\frac{5\pi}{3} - \frac{\pi}{2}\right) = P\left(\frac{7\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	*x*- and *y*-values change places and take signs of new quadrant

Diagrams:

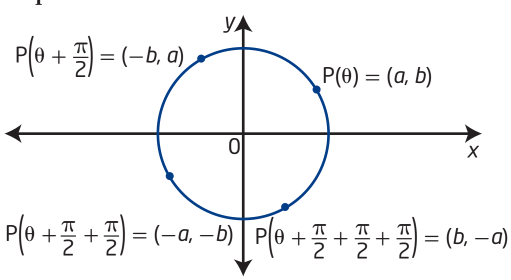
Steps 1–3







Step 4



9. a) $x^2 + y^2 = 1$ **b)** $\left(\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$

c) $\theta + \frac{\pi}{2}$ **d)** quadrant IV

e) maximum value is +1, minimum value is − 1

10. a) Yes. In quadrant I the values of cos θ decrease from 1 at θ = 0° to 0 at θ = 90°, since the *x*-coordinate on the unit circle represents cos θ, in the first quadrant the values of *x* will range from 1 to 0.

b) Substitute the values of *x* and *y* into the equation $x^2 + y^2 = 1$, Mya was not correct, the correct answer is $y = \sqrt{1 - (0.807)^2}$
$= \sqrt{0.348\ 751}$
$\approx 0.590\ 551$

c) $x = 0.9664$

11. b) All denominators are 2.

c) The numerators of the *x*-coordinates decrease from $\sqrt{3}$, $\sqrt{2}$, $\sqrt{1} = 1$, the numerators of the *y*-coordinates increase from $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$. The *x*-coordinates are moving closer to the *y*-axis and therefore decrease in value, whereas the *y*-coordinates are moving further away from the *x*-axis and therefore increase in value.

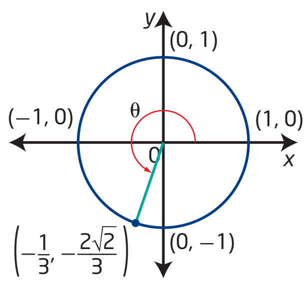
d) Since $x^2 + y^2 = 1$ then $x = \sqrt{1 - y^2}$ and $y = \sqrt{1 - x^2}$, all solutions involve taking square roots.

12. a) $-2\pi \leq \theta < 4\pi$ represents three rotations around the unit circle and includes three coterminal angles for each point on the unit circle.

b) If $P(\theta) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, then $\theta = -\frac{4\pi}{3}$ when $-2\pi \leq \theta \leq 0$, $\theta = \frac{2\pi}{3}$ when $0 \leq \theta \leq 2\pi$, and $\theta = \frac{8\pi}{3}$ when $2\pi \leq \theta < 4\pi$.

c) All these angles are coterminal since they are all 2π radians apart.

13. a)

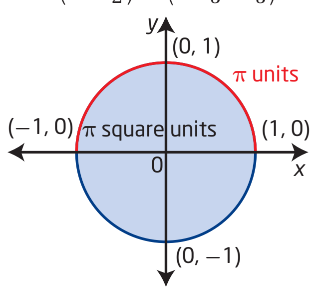


This point represents the terminal point of an angular rotation on the unit circle.

b) quadrant III **c)** $P\left(\theta + \frac{\pi}{2}\right) = \left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$

d) $P\left(\theta - \frac{\pi}{2}\right) = \left(-\frac{2\sqrt{2}}{3}, \frac{1}{3}\right)$

14.



π units is the perimeter of half of a unit circle since $a = r\theta = (1)\pi =$ π units. π square units is the area of a unit circle since $A = \pi r^2 = \pi(1)^2$
$= \pi$ square units.